DISCONTINUOUS BOUNDARY CONDITIONS AND THE DIRICHLET PROBLEM*

BY

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It has been suggested that in the Dirichlet problem there is something essentially antagonistic between the utmost degree of generality attainable as regards the geometrical character of the boundary and the utmost of generality attainable as regards the boundary values assumed over this boundary. The purpose of this paper is to show that the solution of the Dirichlet problem merely for continuous boundary conditions at once entrains the unique determination of a harmonic function correlated with discontinuous boundary conditions of a very general sort. The logical tool employed to this end is the Daniell integral.† In the stress laid on generalized types of integration this paper is closely akin to one of G. C. Evans,‡ but it appears to the author that though the theory of Evans gives more detailed information concerning the character of the solutions of the Dirichlet problem in the neighborhood of the boundary, it is less direct than the present theory, and less extensible to regions of infinite connectivity or higher dimensionality.

I. The potential at a point as a linear functional of the boundary conditions

Let R be any open set of points in n-space connected in the sense that any two of its points form extremities of a polygonal line lying entirely within it, and not extending to infinity. Let the Dirichlet problem be solvable over R for any continuous boundary conditions on C. That is, if $U(P)$ is defined for every point P on C, and if

$$\lim_{\overline{PQ}\to 0} U(P) = U(Q)$$

* Presented to the Society, April 28, 1923.

† *A general form of integral,* Annals of Mathematics, ser. 2, vol. 19 (1918), p. 279.

‡ *Problems of potential theory,* Proceedings of the National Academy of Sciences, vol. 7 (1921), pp. 89–98.

for every Q on C, then there is a function $u(P)$, defined and continuous over $R+C$, harmonic over R, and reducing to U on C. With the aid of a form of generalized integral due to Daniell, we shall associate with any discontinuous function $V(P)$ of a very wide class of functions defined over C a unique function $v(P)$ defined and harmonic over R, which we might naturally call the solution of the boundary value problem corresponding to $V(P)$.

Let Q be a given point in the interior of R. Then $u(Q)$ may be regarded as a functional of $U(P)$. To symbolize this point of view, let us write

$$I_Q(U) = u(Q).$$

Since the functions harmonic over R form a linear set, we have

$$(C) \qquad I_Q(CU) = CI_Q(U),$$

and

$$(A) \qquad I_Q(U_1+U_2) = I_Q(U_1)+I_Q(U_2).$$

Moreover, because of the fact that no function harmonic over R has extrema over R,

$$(P) \qquad I_Q(U) \geqq 0 \quad \text{if} \quad U \geqq 0.$$

Another property of I_Q is that

(L) If $U_1, U_2, \ldots . U_n, \ldots$ is a sequence of continuous functions defined over C, and

$$U_1 \geqq U_2 \geqq \cdots \geqq 0 = \lim U_n$$

for every point on C, then

$$\lim I_Q(U_n) = 0.$$

To prove this, it is sufficient to show that

$$(1) \qquad \lim \max U_n = 0.$$

Since

$$I_Q(U_n) \leqq \max U_n.$$

If (1) were false, it would be possible to find a positive number ϵ and a sequence $\{P_n\}$ of points on C such that for every n

$$U_n(P_n) > \epsilon.$$

This sequence will have at least one limit element N, which will belong to C, as C is a closed, bounded set. By hypothesis

$$U_1(N) \geqq U_2(N) \geqq \cdots \geqq 0 = \lim U_n(N).$$

Hence there is a value of n, say ν, such that

$$(2) \qquad U_\nu(N) < \epsilon/2.$$

Let $\{N_n\}$ be a sequence selected from the P_n's and approaching N as a limit. It follows from the hypothesis of (L) that for every N_n from some stage $n = \mu$ on

$$U_\nu(N_n) > \epsilon.$$

Hence, since U_ν is continuous,

$$(3) \qquad U_\nu(N) \geqq \epsilon.$$

Since formulas (2) and (3) contradict one another, (1) and hence (L) is proved.

The continuous functions defined over C form a linear set: that is, the sum of any two or a constant multiple of any one is also continuous. The absolute value of a continuous function is continuous. A continuous function is bounded over C. These properties of continuous functions are those attributed to a class T_0 by Daniell in the paper to which we have already referred. Moreover (C), (A), (P), and (L) show that the operator I_Q fulfils the conditions which he has laid down for an operator I on T_0. We are hence in a position to employ those definitions and theorems by which he enlarges the scope of the operator I.

II. The Daniell extension of I_Q

In accordance with a definition of Daniell, a function U is of the class T_1 if $U_1 \leqq U_2 \leqq \cdots$ is a non-decreasing sequence of functions from T_0 and $U = \lim U_n$. Under these circumstances, by a theorem of Daniell,

$$I_Q(U_1) \leqq I_Q(U_2) \leqq \cdots.$$

and either $\lim I_Q(U_n)$ exists or else $I_Q(U_n)$ becomes infinite. We shall write

$$(4) \qquad u(Q) = I_Q(U) = \lim I_Q(U_n) = \lim u_n(Q).$$

In general, we shall preserve the usage here indicated for the correlation of corresponding small and capital letters, applying the latter to functions on C, the former to the functions thereby determined as generalized integrals on R.

Now $u_n(Q)$ is harmonic over R. Hence the sequence $\{u_n\}$ is a monotone sequence of harmonic functions. It follows from a theorem of Harnack* that if $u(Q)$ is finite for any Q in R, it is finite and harmonic for every Q in R, while $u_n(Q)$ converges uniformly to $u(Q)$ over any closed region S interior to R.

In accordance with Daniell's definition, if V is any function defined on C, we shall write $\dot{I}_Q(V)$ for the lower bound of $I_Q(U)$ for all the functions U of class T_1 such that $U \geqq V$. Similarly, we make the definition

$$\underset{\cdot}{I}_Q(V) = -\dot{I}_Q(-V).$$

When we have $\underset{\cdot}{I}_Q(V) = \dot{I}_Q(V)$, we say that V is summable, and with Daniell write what is in our notation

$$v(Q) = I_Q(V) = \underset{\cdot}{I}_Q(V) = \dot{I}_Q(V).$$

When V is summable, it is clearly possible to find, whatever positive number ϵ may be, a pair of functions U_1 and U_2, such that U_1 and $-U_2$ belong to T_1, while

$$U_1 \geqq V \geqq U_2,$$

and

$$I_Q(U_1) + I_Q(-U_2) = u_1(Q) - u_2(Q) < \epsilon.$$

The function $u_1(P) - u_2(P)$ is clearly harmonic. We now proceed as in the proof of Harnack's theorem.† Let us describe about Q any circle lying entirely within R. Let $u_1 - u_2$ assume the values W on the periphery of this circle, the radius of which we take to be a. Then over the interior of the circle

* Osgood, *Lehrbuch der Funktionentheorie*, Leipzig, 1907, p. 615. This theorem is there proved for the two-dimensional case, but the proof may be extended at once to n dimensions. The restriction of the theorem to "Bereiche S" is not essential, as it applies at once to all connected open regions.

† The proof is here given in the form appropriate to a space of two dimensions, but mutatis mutandis is valid in n dimensions. For the sake of explicitness the language of two dimensions is used in several subsequent passages.

$$u_1 - u_2 = \frac{1}{2\pi}\int_0^{2\pi} W \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \psi) + r^2}\, d\psi.$$

Since

$$0 \leqq \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \psi) + r^2} \leqq \frac{a+r}{a-r} \qquad (r < a)$$

we have

$$u_1 - u_2 \leqq \frac{1}{2\pi}\int_0^{2\pi} W \frac{a+r}{a-r}\, d\psi$$

$$(5) \qquad = \{u_1(Q) - u_2(Q)\}\frac{a+r}{a-r}$$

$$< \varepsilon \frac{a+r}{a-r}.$$

It follows that within any circle about Q of radius less than a, by making ε approach 0, we shall make $u_1 - u_2$ approach 0 uniformly. Since $u_1(P) \geqq \dot{I}_P(V)$ and $u_2(P) \leqq \underset{\cdot}{I}_P(V)$, it follows that within this circle $v(P)$ exists, and is approached uniformly by $u_1(P)$ and $u_2(P)$ as ε approaches 0. Hence within this circle v is harmonic. By the formation of chains of circles such as those used in the theory of analytic continuation, it may be shown that if V is summable with regard to one point of R, it is summable with regard to all points of R, and that the function $v(P)$ thus defined is harmonic everywhere in R.

Daniell shows that the set of all summable functions is closed with regard to the operations of addition, multiplication by a constant, and taking the absolute value. He also shows that the extended operation I_Q satisfies (C), (A), and (P). Condition (L) is replaced by the important theorem which reads in our terminology as follows: if $\{U_n\}$ is a sequence of summable boundary functions such that $\lim U_n = U$, and if there is a summable function V such that over all C we have $|U_n| \leqq V$ for all n, then U is summable, and $u(P) = \lim u_n(P)$. As one consequence of these theorems, functions belonging to T_0 and T_1 are summable, and all the definitions given of $I_Q(U)$ agree when more than one of them is applicable.

III. The behavior of $I_Q(U)$ in the neighborhood of the boundary

If Q is a point of C, and K is a circle with Q as center, then any function $U(P)$ which is defined and summable on C, bounded, and 0 over that part of C

within K, determines a harmonic function $u(P)$ which approaches 0 as P approaches Q. To prove this, let us take a as the radius of K, and let K' be a concentric circle with radius $a/2$. Let the upper bound of $|U|$ be M. Then we shall define the function $V(P)$ as M outside of K, 0 within K', and $M((2r/a)-1)$ at those points in the annulus between K and K' at a distance r from Q. Clearly V is continuous. The boundary values V on C determine a function $v(P)$ continuous on $R+C$, harmonic on R, and reducing to V on C. By definition

$$|U(P)| \leqq V(P),$$

over C. Hence by a theorem of Daniell

$$|u(P)| \leqq I_P(|U|) \leqq v(P).$$

This holds over all R. Since

$$\lim_{P\to Q} v(P) = 0 = V(Q),$$

we have

$$\lim_{P\to Q} u(P) = 0 = U(Q).$$

This result is capable of immediate generalization. In the first place, by the addition of a constant to U, we get the result that if $W(P)$ is a bounded function summable on C and constant over K.

$$\lim_{P\to Q} w(P) = W(Q). \tag{6}$$

Next let W_1 be any bounded summable function defined over C, and let A and B be respectively the upper and lower bounds of the oscillation of W_1 in the neighborhood of the point Q of C. Then given any positive quantity ϵ, it is possible to describe a circle K about Q as center within which $A+\epsilon > W_1(P) > B-\epsilon$. It then follows from the extension of (P) to all summable functions and from (6) that if the functions formed from W_1 by substituting $A+\epsilon$ or $B-\epsilon$ within K are summable, the oscillation of $w_1(P)$ in the neighborhood of Q is not greater than from $A+\epsilon$ to $B-\epsilon$, and hence, since ϵ is arbitrary, is not greater than from A to B. If in particular W_1 is continuous at Q, then the function which is W_1 on C and w_1 on R is likewise continuous there.

Daniell has a theorem to the effect that a function equal for every argument to the greater or to the less of two summable functions is itself summable. The function which is the constant H over the part of C within K and 0 elsewhere may be shown without difficulty to belong to T_1, and hence to be summable. Hence the function obtained from W_1 by replacing its values within K by $A+\epsilon$ or $B-\epsilon$ is in fact summable.

I have so far been unable to eliminate without any further restriction the condition that W_1 be bounded, although I suspect that this condition is superfluous. The results obtained in this section are valid, of course, if for the word "circle" is substituted "n-sphere", in a space of n dimensions.

IV. Certain summable functions

A surface M in n-space is said to have capacity c if there is a positive function $f(P)$ defined over M such that

$$1 = \left(\int_M\right)^{n-1} f(P)\,(PQ)^{n-2}\,dS$$

(in the case $n=2$,

$$1 = -\int_M f(P)\log PQ\,dS)$$

independently of Q so long as Q remains on M, while

$$\left(\int_M\right)^{n-1} f(P)\,dS = c.$$

A set of points on C is said to have zero measure if the function which is 1 over the set and 0 over the rest of C is summable, and determines a harmonic function 0 over R. I say that if we have given a closed set of points S on the boundary C of an n-dimensional region R for which the Dirichlet problem is solvable, and if S can be included in the interior of a surface M of arbitrarily small capacity, then S is of zero measure.

To prove this, let us remark that we can distribute over the part of C within M a boundary potential that is continuous, 1 on S, and 0 on the boundary of M. This boundary potential, together with the boundary potential 0 over the rest of C, will determine a harmonic function $u(Q)$ over R, which will be uniformly less, within R and outside M, than

$$\left(\int_M\right)^{n-1} f(P)\,(PQ)^{n-2}\,dS. \tag{7}$$

or its two-dimensional analogue.* Now let Q be a point of R at a distance a from the nearest point of C. Then (7) and hence $u(Q)$ will not exceed ca^{n-2} or $-c\log a$, as the case may be. Since C may be made arbitrarily small, the $\dot{I}_Q$ of our boundary function that is 0 except on S, where it is 1, cannot exceed 0. It follows at once that S is of zero measure. Thus a finite set of points, or, more generally, an $(n-2)$-spread on the $(n-1)$-spread bounding a region in the space of n dimensions is of zero measure.

A function which is bounded and has discontinuities only at a closed set of points of zero measure is summable. Let S be this set. Let M_a be the set of all points on C within a distance of less than a from some point of S. Let $U(P)$ be any function bounded on C and with all its discontinuities on S. It is then possible to form a function $U_a(P)$ continuous over the whole of C and differing from $U(P)$ only over M_a.† It is even possible to keep the set of functions $\{U_a\}$ uniformly bounded. These functions U_a remain summable even if any finite change is made in them over S, since any function finite over S and zero over the rest of C is less in absolute value than a function with Daniell integral everywhere zero, and hence is itself of zero Daniell integral. As a approaches 0, these functions U_a thus modified can be made to approach U boundedly over all C. Hence by a theorem of Daniell which we have already quoted, U_a is summable.

* In the two-dimensional case, to avoid difficulties due to the change of sign of the logarithm, the unit of measurement should be chosen greater than the greatest linear dimension of C.

† Given any set of points, it is possible to surround it within any assigned distance by a polyhedral boundary. It is hence possible to surround the set by a sequence of such boundaries, each including the next, approaching to the set within any assigned distance. Given any function uniformly continuous on the original set, it is possible to approximate to it by continuous functions on the polyhedral boundaries which all lie between the bounds of the original function, and which all constitute a function continuous over the set of points consisting of the original set, together with the polyhedra. The continuous functions thus obtained may be taken as the boundary values of an infinite set of harmonic functions defined between the successive polyhedra and outside the outermost one. Together these constitute a continuous function extending the original continuous function through the whole of space, and a fortiori over any surface in space.

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A TYPE OF DIFFERENTIAL SYSTEM CONTAINING A PARAMETER*

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Poincaré† and others have discussed the continuity with respect to a parameter μ of the solutions of a system of equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n, \mu, t),$$

in which the function X_i is analytic in μ if $|\mu| < c$, and satisfies certain other conditions with respect to $x_1, x_2, \ldots, x_n, t$ in a domain

$$|x_i - x_i^0| < b, \qquad 0 \leqq t \leqq T \qquad (i = 1, 2, \ldots, n).$$

For certain problems in mechanics it is convenient to have a similar discussion of equations of the form

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n, \cos\nu t, \sin\nu t) \qquad (i = 1, 2, \ldots, n)$$

for very large values of the parameter ν; the present paper is devoted to this type of equations, and an application is made to a problem related to the restricted problem of three bodies.

1. Integration by the method of successive approximations

In the equations

$$(1) \qquad \frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n, \cos\nu t, \sin\nu t)$$

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† *Les Méthodes nouvelles de la Mécanique céleste,* vol. I, § 27.

assume that the functions X_i are continuous in all their arguments, and satisfy the conditions

$$
\begin{aligned}
&|X_i(x_1'', \ldots, x_n'', \cos\nu t, \sin\nu t) - X_i(x_1', \ldots, x_n', \cos\nu t, \sin\nu t)| \\
(2) \qquad &< A_1|x_1'' - x_1'| + \cdots + A_n|x_n'' - x_n'|, \\
&|X_i| < M \qquad (i = 1, 2, \ldots, n),
\end{aligned}
$$

if the arguments lie in the domain

$$
(D) \qquad |x_i - x_i^0| < b, \quad -\infty < t < \infty \quad (i = 1, 2, \ldots, n).
$$

Then if $K = \sum_{i=1}^{n} A_i$, equations (1) can be integrated by the method of successive approximations,* and the solutions are defined if

$$
(3) \qquad 0 \leqq t - t_0 < \frac{1}{K}\log\left(1 + \frac{bK}{M}\right).
$$

In the discussion of the successive approximations we shall require the following

LEMMA. *Assume that* $x_1(t), \ldots, x_n(t)$ *are continuous, possess continuous derivatives, and that* $|x_i - x_i^0| < b$ *for* $0 \leqq t \leqq T$, *and suppose* $y_i(t)$, $Y_i(t)$ *defined by the equations*

$$
\begin{aligned}
&Y_i(t) = \int_0^t X_i(x_1, \ldots, x_n, \cos\nu t, \sin\nu t)\,dt, \\
(4) \qquad & \\
&y_i(t) = \int_0^t \frac{1}{2\pi}\int_0^{2\pi} X_i(x_1, \ldots, x_n, \cos\omega, \sin\omega)\,d\omega\,dt.
\end{aligned}
$$

* Picard, *Traité d' Analyse*, vol. 2, 1905, Chap. XI.

Then if ε is an arbitrarily small positive quantity, N can be found such that if $\nu > N$,

$$|Y_i(t) - y_i(t)| < \varepsilon, \qquad 0 \leqq t \leqq T.$$

If $\Delta t = \dfrac{2\pi}{\nu}$,

$$\cos\nu(t+\Delta t) = \cos\nu t, \quad \sin\nu(t+\Delta t) = \sin\nu t.$$

Suppose

$$m\Delta t \leqq t' < (m+1)\Delta t, \qquad m\Delta t = t.$$

Then

$$(5) \qquad \begin{aligned} Y_i(t') = \sum_{k=1}^{m} \int_{(k-1)\Delta t}^{k\Delta t} X_i(x_1, x_2, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt \\ + \int_{t}^{t'} X_i(x_1, x_2, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt. \end{aligned}$$

Since $\nu(k-1)\Delta t = 2(k-1)\pi$, $\nu k\Delta t = 2k\pi$,

$$I_k = \int_{(k-1)\Delta t}^{k\Delta t} X_i(x_1, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt$$

$$= \frac{1}{\nu}\int_0^{2\pi} X_i(x_1, \ldots x_n, \cos\omega, \sin\omega)\, d\omega$$

if

$$\omega = \nu[t-(k-1)\Delta t],$$

or finally

$$(6) \qquad I_k = \frac{\Delta t}{2\pi}\int_0^{2\pi} X_i(x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \cos\omega, \sin\omega)\, d\omega.$$

where $x_i^{(k-1)}$ denotes the function $x_i(t)$ in the interval $(k-1)\Delta t \leqq t \leqq k\Delta t$, or the corresponding function of ω. If $\bar{x}_i^{(k-1)} = x_i[(k-1)\Delta t]$, we obtain from (2)

$$|X_i(x_1^{(k-1)}, \ldots, x_n^{(k-1)}, \cos\omega, \sin\omega) - X_i(\bar{x}_1^{(k-1)}, \ldots. \bar{x}_n^{(k-1)}, \cos\omega, \sin\omega)|$$

$$< \sum_{r=1}^{n} A_r \, |x_r^{(k-1)} - \bar{x}_r^{(k-1)}|.$$

Since $x_i'(t)$ exists and is continuous for $0 \leqq t \leqq T$, we have, for some constant B,

$$|x_i'(t)| < B, \qquad 0 \leqq t \leqq T.$$

Then

$$|x_i^{(k-1)} - \bar{x}_i^{(k-1)}| < B\Delta t,$$

(7) $$\sum_{i=1}^{n} A_i |x_i^{(k-1)} - \bar{x}_i^{(k-1)}| < BK\Delta t.$$

If

$$I_k' = \frac{\Delta t}{2\pi} \int_0^{2\pi} X_i(\bar{x}_1^{(k-1)}, \ldots, \bar{x}_n^{(k-1)}, \cos\omega, \sin\omega)\, d\omega,$$

$$|I_k - I_k'| \leqq \frac{\Delta t}{2\pi} \int_0^{2\pi} BK\Delta t \, d\omega$$

(8) $$\leqq BK(\Delta t)^2.$$

Also

(9) $$\left| \int_t^{t'} X_i(x_1, \ldots, x_n, \cos\nu t, \sin\nu t)\, dt \right| < M\Delta t.$$

From (5), (8), (9),

$$\left| Y_i(t') - \sum_{k=1}^{m} I'_k \right| \leqq M\Delta t + BKm(\Delta t)^2$$

$$\leqq \Delta t(M + BK\bar{t}). \tag{10}$$

From the definition of $y_i(t)$,

$$y_i(t) = \int_0^t \frac{1}{2\pi} \int_0^{2\pi} X_i(x_1, x_2, \ldots, x_n, \cos\omega, \sin\omega)\, d\omega\, dt$$

$$= \sum_{k=1}^{m} \frac{\Delta t}{2\pi} \int_0^{2\pi} X_i\left[x_1^{(k-1)}(\xi), x_2^{(k-1)}(\xi), \ldots, x_n^{(k-1)}(\xi), \cos\omega, \sin\omega\right] d\omega,$$

$$(k-1)\Delta t \leqq \xi \leqq k\Delta t,$$

from the theorem of the mean. Also,

$$|y_i(\bar{t}) - y_i(t')| < M\Delta t.$$

Employing inequalities (2),

$$\left| y_i(\bar{t}) - \sum_{k=1}^{m} I'_k \right| \leqq BK\bar{t}\Delta t.$$

Hence, finally,

$$\left| y_i(t') - \sum_{k=1}^{m} I'_k \right| \leqq \Delta t[M + BK\bar{t}]. \tag{11}$$

Combining (10) and (11),

$$(12)\qquad |Y_i(t') - y_i(t')| \leqq \frac{4\pi}{\nu}[M + BKT], \quad 0 \leqq t' \leqq T.$$

From (12) the lemma follows immediately.

Equations (1) can be integrated by the construction of the functions

$$(13)\quad u_i^{(k)} = x_i^0 + \int_0^t X_i(u_1^{(k-1)}, u_2^{(k-1)}, \ldots, u_n^{(k-1)}, \cos\nu t, \sin\nu t)\,dt, \; u_i^0 = x_i^0$$

$$(i = 1, 2, \ldots, n;\; k = 1, 2, \ldots).$$

Consider the differential equations

$$(14)\quad \frac{dz_i}{dt} = \frac{1}{2\pi}\int_0^{2\pi} X_i(z_1, z_2, \ldots, z_n, \cos\omega, \sin\omega)\,d\omega \quad (i = 1, 2, \ldots, n),$$

and suppose these integrated by the same method;

$$(15)\quad v_i^{(k)} = x_i^0 + \int_0^t \frac{1}{2\pi}\int_0^{2\pi} X_i(v_1^{(k-1)}, v_2^{(k-1)}, \ldots, v_n^{(k-1)}, \cos\omega, \sin\omega)\,d\omega\,dt,$$

$$v_i^0 = x_i^0, \qquad (i = 1, 2, \ldots, n;\; k = 1, 2, \ldots).$$

It is seen immediately that the same constants K, M can be employed for equations (14). Also the functions $u_i^{(k)}$, $v_i^{(k)}$ lie in the domain D.

From (13)

$$u_i^{(k)} = x_i^0 + \int_0^t X_i(v_1^{(k-1)}, v_2^{(k-1)}, \ldots, v_n^{(k-1)}, \cos\nu t, \sin\nu t)\,dt$$

$$(16)\qquad + \int_0^t [X_i(u_1^{(k-1)}, \ldots, u_n^{(k-1)}, \cos\nu t, \sin\nu t)$$

$$- X_i(v_1^{(k-1)}, \ldots, v_n^{(k-1)}, \cos\nu t, \sin\nu t)]\,dt.$$

Suppose $|u_i^{(k-1)} - v_i^{(k-1)}| < \varrho$ $(i = 1, 2, \ldots, n)$; then if $w_i^{(k)}$ is defined by the equation

$$w_i^{(k)} = x_i^0 + \int_0^t X_i(v_1^{(k-1)}, \ldots, v_n^{(k-1)}, \cos\nu t, \sin\nu t)\,dt,$$

$$|u_i^{(k)} - w_i^{(k)}| < K\varrho t \qquad (i = 1, 2, \ldots, n).$$

Also, from the lemma,

$$|w_i^{(k)} - v_i^{(k)}| \leqq \frac{4\pi}{\nu}[M + BKT].$$

Hence

$$|u_i^{(k)} - v_i^{(k)}| \leqq K\varrho t + \frac{4\pi}{\nu}[M + BKT].$$

From the definition of the functions u, v, w it follows that B can be replaced by M.

If $k = 1$, we obtain

$$|u_i' - v_i'| \leqq \frac{4\pi M}{\nu}[1 + KT] = \frac{C_1}{\nu}.$$

Similarly,

$$|u_i'' - v_i''| \leqq \frac{KTC_1}{\nu} + \frac{C_1}{\nu} = \frac{C_2}{\nu},$$

and in general

(17) $$|u_i^{(k)} - v_i^{(k)}| \leqq \frac{C_k}{\nu} \qquad (k = 1, 2, \ldots),$$

where

$$C_k = 4\pi M(1 + KT)\sum_{r=0}^{k-1}(KT)^r.$$

Now if $T < \frac{1}{K}\log\left(1+\frac{bK}{M}\right)$, the integer ι can be chosen so large that if $\sigma > \iota$, $0 < t \leqq T$,

$$|x_i(t) - u_i^{(\sigma)}(t)| < \frac{\varepsilon}{3},$$

$$|z_i(t) - v_i^{(\sigma)}(t)| < \frac{\varepsilon}{3},$$

if ε is any previously assigned positive quantity.

Now suppose σ fixed; from (17)

$$|u_i^{(\sigma)} - v_i^{(\sigma)}| \leqq \frac{C_\sigma}{\nu}.$$

Hence if ν is chosen sufficiently large $\frac{C_\sigma}{\nu} < \frac{\varepsilon}{3}$, and we obtain

$$|x_i(t) - z_i(t)| < \varepsilon \qquad (i = 1, 2, \ldots, n).$$

Hence the theorem: *If* $x_1(t), x_2(t), \ldots, x_n(t)$ *is a solution of* (1), *and* $z_1(t), z_2(t), \ldots, z_n(t)$ *the solution of* (14) *satisfying the same initial conditions, and if* $0 < T < \frac{1}{K}\log\left(1+\frac{bK}{M}\right)$, *then given any positive quantity* ε, *a number* N *can be found such that if* $\nu > N$,

$$|x_i(t) - z_i(t)| < \varepsilon \qquad (0 \leqq t \leqq T;\ i = 1, 2, \ldots, n).$$

If $KT < 1$, then $C_\sigma < C$.

$$C = \frac{4\pi M(1+KT)}{1-KT}.$$

Hence for any ν, ι can be so chosen that if $\sigma > \iota$,

$$|x_i(t) - u_i^{(\sigma)}(t)| < \frac{1}{2\nu}, \qquad |z_i(t) - v_i^{(\sigma)}(t)| < \frac{1}{2\nu}, \qquad |u_i^{(\sigma)} - v_i^{(\sigma)}| < \frac{C}{\nu}.$$

Consequently

$$|x_i(t) - z_i(t)| < \frac{C+1}{\nu} \qquad (0 \leqq t \leqq T;\ i = 1, \ldots, n),$$

an inequality independent of ι.

2. Example

Suppose μ a parameter on the interval $0 < \mu < 1$, and assume masses μ $1-\mu$ connected by a rigid weightless bar of unit length. Assume this system to rotate about its center of gravity in the (x, y) plane, with an angular velocity n, the origin coinciding with the center of gravity. Then if a particle of unit mass moves in space under the newtonian attraction of the first two masses, its coördinates satisfy the equations

$$\begin{aligned}
&\frac{dx}{dt} = x', \qquad \frac{dx'}{dt} = \frac{\partial U}{\partial x},\\
&\frac{dy}{dt} = y', \qquad \frac{dy'}{dt} = \frac{\partial U}{\partial y},\\
&\frac{dz}{dt} = z', \qquad \frac{dz'}{dt} = \frac{\partial U}{\partial z},\\
&U = \frac{1-\mu}{r_1} + \frac{\mu}{r_2},\\
&x_1 = \mu \cos nt, \qquad y_1 = \mu \sin nt,\\
&x_2 = (1-\mu)\cos nt, \qquad y_2 = -(1-\mu)\sin nt,\\
&r_1^2 = (x-x_1)^2 + (y-y_1)^2 + z^2,\\
&r_2^2 = (x-x_2)^2 + (y-y_2) + z^2.
\end{aligned} \tag{18}$$

If $P_0(x_0, y_0, z_0)$ is such that $r_1^0 > 1$, $r_2^0 > 1$, or if $z_0 \neq 0$, then a certain neighborhood of P_0 can be found within which the first and second partial derivatives of U, with respect to x, y, z, are continuous and their absolute values have upper bounds independent of n. Within this neighborhood equations (18) are of the form (1), and the theorem of § 1 can be applied. The motion approaches that defined by the equations

$$
\begin{aligned}
&\frac{d^2x}{dt^2} = \frac{\partial \overline{U}}{\partial x}, \\
&\frac{d^2y}{dt^2} = \frac{\partial \overline{U}}{\partial y}, \qquad \overline{U} = \frac{1-\mu}{2\pi}\int_0^{2\pi}\frac{d\omega}{\overline{r}_1} + \frac{\mu}{2\pi}\int_0^{2\pi}\frac{d\omega}{\overline{r}_2}, \\
&\frac{d^2z}{dt^2} = \frac{\partial \overline{U}}{\partial z}, \\
&\overline{x}_1 = \mu\cos\omega, \qquad \overline{y}_1 = \mu\sin\omega, \\
&\overline{x}_2 = (1-\mu)\cos\omega, \qquad \overline{y}_2 = -(1-\mu)\sin\omega.
\end{aligned} \tag{19}
$$

The limiting motion is that of a particle moving in space under the attraction of two concentric rings, each of uniform density; the equations (19) admit the area integral

$$xy' - x'y = C,$$

in addition to the energy integral. Consequently the plane problem is integrable.

The interest of this result lies in the fact that while in the restricted problem of three bodies $n = 1$, yet the analytic discussion in many cases* is precisely the same as for n arbitrary ($\neq 0$).

* For instance, Birkhoff, *The restricted problem of three bodies*, Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915).

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ON A REMARKABLE CLASS OF ENTIRE FUNCTIONS*

BY

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The problem treated in the following pages was first studied by Laguerre† in this form. Let

$$P_n(x) = A_{0n} + A_{1n}x + A_{2n}x^2 + \cdots + A_{nn}x^n \quad (n = 1, 2, \ldots)$$

represent a sequence of polynomials which converges towards an entire function $F(x)$ as a limit. Suppose, further, that, for each value of n, $P_n(x)$ has all its roots real. Laguerre proved that $F(x)$ can be expressed as a canonical product, of genus not greater than 1, multiplied by an exponential of the form e^{ax^2+bx+c}. If the additional assumption be made that all the roots of all the polynomials have the same sign (either all positive, or all negative) then the canonical product is of genus 0 and $a = 0$.

In 1907 Petrovitch‡ took up the problem in a more restricted form by assuming that the polynomials $P_n(x)$ are the sections of the power series for the limiting function. That is, if we write

$$F(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots, \tag{1}$$

then

$$P_n(x) = F_n(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

The coefficients then satisfy the necessary inequalities

$$(n-1)\, a_{n-1}^2 - 2n\, a_n\, a_{n-2} \geqq 0. \tag{2}$$

The conditions that $F_n(x)$ and $F(x)$ shall have all their roots real are given by Petrovitch in this way. Let $\Delta(a_0, a_1, \ldots, a_{n-1}, a_n)$ be the discriminant of $F_n(x)$ with $a_0, a_1, \ldots, a_{n-1}$ given and a_n to be determined. Then a_n

* Presented to the Society, April 28, 1923.

† *Sur les fonctions de genre un ou de genre zéro*, Oeuvres I, p. 174.

‡ *Une classe remarquable de séries entières, Atti del* IV *Congresso internationale dei Matematici*, Rome, 1908, pp. 36–43.

can have any value between the least negative and the least positive root of the equation $\Delta = 0$ with a_n regarded as the unknown. We have here an infinity of conditions for the step by step determination of the limits between which $a_2, a_3, \ldots,$ can be arbitrarily chosen, a_0 and a_1 being unrestricted. These conditions being in a form impossible to use for the effective construction of such series, Petrovitch refers to a paper by Mr. E. G. Hardy* for definite examples of series having the required property and then shows how, according to Laguerre, we may operate on any of these to obtain additional series in unlimited number.

After these general results, the remainder of Petrovitch's paper is devoted to a detailed study of the case $a_n \geqq 0$, $n = 0, 1, \ldots,$ so that all the roots of $F_n(x)$ and of $F(x)$ are of the same (negative) sign. Quoting Laguerre he erroneously concludes that $F(x)$ is of the form mentioned above without observing that in his restricted case b (as well as a) is necessarily zero. In fact, as Petrovitch shows, the conditions (2) are equivalent to

$$\sqrt[n]{a_n} < \frac{e\,a_1}{(n+1)(\sqrt{2})^n}.$$

From these conditions we may at once conclude from a well known theorem† that *the genus of all functions of the Petrovitch class whose roots are all of the same sign is zero.*

The chief object of the present paper is to extend the results obtained by Hardy. We assume, with Hardy, that the coefficients a_n in (1) are of the form

$$a_n = \frac{1}{b_1 b_2 \ldots b_n}, \tag{3}$$

the b_n being all positive and $a_0 = 1$ for convenience. It should be observed that the coefficients of any series may be represented in the form (3). If $\lim_{n \to \infty} b_n = L$, then the radius of convergence is L. In order to exclude all but entire functions from consideration, we assume $L = \infty$.

It is remarkable that, while the conditions on the a_n, as given by Petrovitch, are impossible of solution, the form given to the coefficients by Hardy enables

* *On the zeros of a class of integral functions*, Messenger of Mathematics, vol. 34 (1904), pp. 97–101.

† See, for example, E. Lindelöf, *Mémoire sur la théorie des fonctions entières*, Acta Societatis Scientiarum Fennicae, vol. 31, Art. 20, p. 46.

the required conditions to be put in an extraordinarily simple form, requiring, however, a slight restriction upon the complete generality of Petrovitch's problem.

Hardy assumes $b_n \geqq 9 b_{n-1}$ and proves that the function $f(x) = 1 + t_1 + t_2 + \ldots + t_n + \ldots$, $t_n = a_n x^n$, can be put in the form $t_n(1 + \Phi_n)$, $|\Phi_n| < 1$, on certain circles $|x| = r_n$, $n = 1, 2, \ldots$, and hence $f(x)$ has exactly n zeros inside of the circle $|x| = r_n$. These zeros are then proved to be real and negative.

We will now assume the more general condition

$$b_n \geqq \alpha b_{n-1}, \qquad \alpha \text{ real and positive.} \tag{4}$$

Hardy's method is applicable if $\alpha \geqq 4.8106$,* but is useless for smaller values of α. We accordingly adopt a different method which will apply to all cases to which condition (4) is applicable and which will possess the additional advantage of being extremely simple and elementary in character. It will be found that the least value α can have is 4, since in that case we are able to prove the following theorem.

A. *The relations*

$$b_n \geqq 4 b_{n-1} \qquad (n = 2, 3, \ldots), \tag{5}$$

are the necessary and sufficient conditions that the series $f(x)$ may have the properties:

1. *The roots of $f(x)$ are all real, simple, and negative.*

2. *The roots of any polynomial $a_m x^m + \cdots + a_n x^n$ formed by taking any number of consecutive terms of $f(x)$ are all real, simple, and negative* (*excepting* $x = 0$).

3. As a special case of number 2, *any series $f_n(x)$ formed by taking the first $n+1$ terms of $f(x)$ has all its roots real, simple, and negative.*

Introduce, for convenience, the notation

$$\pi_n = b_1 b_2 \cdots b_n,$$

whence

$$f(x) = 1 + \frac{x}{\pi_1} + \frac{x^2}{\pi_2} + \cdots + \frac{x^n}{\pi_n} + \cdots = 1 + t_1 + \cdots + t_n + \cdots.$$

We observe that formula (5) is the necessary and sufficient condition that the polynomial $t_{n-2} + t_{n-1} + t_n$, formed by any three consecutive terms of $f(x)$, shall have real roots. Hence (5) is a necessary condition for Theorem A.

* This is (approximately) the lower limit of α for which $|\Phi_n| < 1$.

To prove (5) a sufficient condition in all cases, we proceed to locate the roots of $f(x)$ and of $f_n(x)$. Give to x the values

$$x = -\sqrt{b_{2\nu-1}\,b_{2\nu}} \qquad (\nu = 1, 2, \ldots). \tag{6}$$

On account of the relations

$$t_n = t_{n-1}\cdot\frac{x}{b_n}, \qquad \frac{x}{b_n} = -\sqrt{\frac{b_{2\nu-1}}{b_n}\cdot\frac{b_{2\nu}}{b_n}},$$

the terms in $f(x)$ will be alternately positive and negative and increasing numerically, as long as $n \leqq 2\nu - 1$, until we reach the numerically largest term, which is $\dfrac{x^{2\nu-1}}{\pi_{2\nu-1}}$. Consider the sum of the largest term and the two adjacent terms, viz.,

$$\frac{x^{2\nu-2}}{\pi_{2\nu-2}} + \frac{x^{2\nu-1}}{\pi_{2\nu-1}} + \frac{x^{2\nu}}{\pi_{2\nu}} = T.$$

The first and last terms are equal, for the given value of x, and hence

$$T = \frac{x^{2\nu-2}}{\pi_{2\nu-2}}\left(2-\sqrt{\frac{b_{2\nu}}{b_{2\nu-1}}}\right) \leqq 0.$$

from (5). Accordingly, when x has any one of the values (6), $f(x)$ may be written

$$f(x) = (1+t_1)+(t_2+t_3)+\cdots+(t_{2\nu-4}+t_{2\nu-3})+T+(t_{2\nu+1}+\cdots) < 0.$$

The inequality follows from the fact that each group in parenthesis is negative. The last group, in particular, being the remainder of the series and consisting of terms alternately negative and positive and steadily decreasing numerically, is also negative.

In a similar manner, when x has the values

$$x = -\sqrt{b_{2\nu}\,b_{2\nu+1}} \qquad (\nu = 1, 2, \ldots), \tag{7}$$

$f(x)$ may be arranged in the form

$$f(x) = 1 + (t_1 + t_2) + \cdots + (t_{2\nu-3} + t_{2\nu-2})$$

$$+ t_{2\nu-1}\left(2 - \sqrt{\frac{b_{2\nu+1}}{b_{2\nu}}}\right) + (t_{2\nu+2} + \cdots) > 0,$$

each group of which is positive, and hence the inequality. We have thus proved the theorem:

B. The series $f(x)$ has an infinity of real, negative zeros. An odd number of these zeros are situated between any two consecutive numbers of the series

$$(8) \qquad 0, \quad -\sqrt{b_1 b_2}, \quad -\sqrt{b_2 b_3}, \ldots, \quad -\sqrt{b_{n-1} b_n}, \quad -\sqrt{b_n b_{n+1}}, \ldots .$$

It remains to prove that *the zeros thus determined are all simple,* that $f(x)$ *has no other zeros, real or imaginary,* and that *just one root of $f(x)$ occurs between any two consecutive numbers of the series* (8). This is done by showing that the polynomials $f_n(x)$ have only real and simple zeros which are separated by the first $n+1$ numbers of the series (8). We proceed exactly as with $f(x)$ to substitute for x the series of numbers (6), or (7), and to group the terms into negative, or positive groups, the only difference being that, when $2\nu > n$, $f_n(x)$ will preserve an invariable sign which will be that of $(-1)^n$. Hence the result:

C. $f_n(x)$ has n real, distinct, and negative roots which are separated by the numbers

$$0, \quad -\sqrt{b_1 b_2}, \quad -\sqrt{b_2 b_3}, \ldots, \quad -\sqrt{b_n b_{n+1}}.$$

We prove at the same time, by this method, that *the polynomial*

$$a_n x^n + a_{n+1} x^{n+1} + \cdots + a_p x^p$$

formed by any number of consecutive terms of $f(x)$ has all its roots real, distinct, and negative (*except for a multiple root $x = 0$*), since, if we remove the factor $a_n x^n$, the coefficients of the resulting polynomial will again be of the form (3), and will satisfy the required conditions (5).

Since $f(x)$ is the limit of $f_n(x)$, the conclusion, stated above, is proved.

Example 1. By specializing condition (4), restricted cases of particular interest may be obtained. The most obvious procedure would be to drop the inequality sign and replace α by a chosen function of n of such a nature as to satisfy (5) for all values of n. As a first example, let α be constant so that we have $b_n = \alpha b_{n-1}$, $n = 1, 2, \ldots$. $b_0 = 1$, whence $b_n = \alpha^n$. The resulting series is

$$\varphi(x) = 1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^3} + \cdots + \frac{x^n}{\alpha^{n(n+1)/2}} + \cdots. \tag{9}$$

Substitute $x = -\alpha^\nu$, $\nu = 1, 2, \ldots$. The $(n+1)$th term becomes

$$(-1)^n \alpha^{n\nu - n(n+1)/2}.$$

Assume $n\nu - \dfrac{n(n+1)}{2} = 0$, whence $n = 2\nu - 1$, and accordingly the $(n+1)$th term reduces to (-1). It is easy to verify that all the first $(n+1)$ terms of $\varphi(x)$ cancel each other, the pth power of x cancelling the $(2\nu - 1 - p)$th power, and hence $\varphi(x)$ reduces to

$$\varphi(-\alpha^\nu) = \frac{1}{\alpha^\nu} - \frac{1}{\alpha^{2\nu+1}} + \cdots.$$

Since $\alpha \geqq 4$, it follows that $\varphi(-\alpha^\nu)$ has a very small positive value less than $\alpha^{-\nu}$ which approaches zero as ν increases indefinitely. In other words, *the roots of $\varphi(x)$ are represented approximately and asymptotically by the series of numbers $-\alpha, -\alpha^2, \ldots, -\alpha^\nu, \ldots$.* This result is evidently true if α has any real or imaginary value such that $|\alpha| > 1$.

The function $\varphi(x)$ satisfies the functional equation

$$\varphi(\alpha x) - x\varphi(x) = 1.^* \tag{10}$$

Denoting the function (9) by $\varphi_1(x)$, we observe that the function

$$\varphi_2(x) = -\frac{1}{x} - \frac{1}{\alpha x^2} - \frac{1}{\alpha^3 x^3} - \cdots - \frac{1}{\alpha^{n(n-1)/2} x^n} - \cdots$$

* This relation was brought to my attention by Professor W. A. Hurwitz.

also satisfies (10). The most general solution of (10) would then be given by the formula

$$\varphi(x) = c\,\varphi_1(x) + (1-c)\,\varphi_2(x),$$

in which c is a constant, or a function of x satisfying the functional equation $c(\alpha x) = c(x)$.

Example 2. Assume $b_n = a^n - 1$, $n = 1, 2, \ldots$, that is, assume

$$b_n = \alpha_n\, b_{n-1}, \qquad \alpha_n = \frac{a^n - 1}{a^{n-1} - 1}.$$

We then obtain

$$\begin{aligned} f(x) &= 1 + \frac{x}{a-1} + \frac{x^2}{(a-1)(a^2-1)} \\ &\quad + \cdots + \frac{x^n}{(a-1)(a^2-1)\cdots(a^n-1)} + \cdots \\ &= \left(1+\frac{x}{a}\right)\left(1+\frac{x}{a^2}\right)\cdots\left(1+\frac{x}{a^n}\right)\cdots, \end{aligned}$$

both expressions being convergent for all values of x if $|a|>1$. If $|a|<1$, the series will be convergent for $|x|<1$. The zeros of this function are evident. When a is real and positive, $f(x)$ will belong to the class designated in Theorem A, if $a \geqq 4$, since $\dfrac{a^n - 1}{a^{n-1}-1} > a$.

If property no. 2 be omitted from Theorem A, the b's may have values somewhat smaller than those determined by (5), as Petrovitch shows by direct calculation. If we assume $b_n = \alpha_n\, b_{n-1}$, α_n being an unknown function of n whose values for $n = 2, 3, \ldots$ are the smallest possible for which Theorem A 1 and 3 will be satisfied, then the results as far as known are: $\alpha_2 = 4$, $\alpha_3 = 3.375$, $\alpha_4 = 3.264$. It would seem that α_n is decreasing steadily to some unknown limit. All that we can say about such a limit is that it must be $\geqq 2$. For the coefficients in (1), as Petrovitch has pointed out, must satisfy (2), which is the necessary and sufficient condition that the $(n-2)$nd derivative of $f_n(x)$ shall have real roots. Expressing the a's in terms of the b's by means of (3), formula (2) reduces to the simpler expression

$$b_n \geqq \frac{2n}{n-1}\, b_{n-1}. \tag{11}$$

Comparing with the values given above for α_n, we see that the equality sign is allowable in (11) for $n = 2$, but for $n = 3$ or 4 the inequality sign must be used. So that for all cases,

$$\alpha_n \geqq \frac{2n}{n-1}, \text{ and hence } \lim_{n \to \infty} \alpha_n \geqq 2.$$

Denoting the moduli of the roots of $f(x)$ by $r_1, r_2, \ldots . r_n, \ldots .$ we have from (7)

$$r_n > \sqrt{b_{n-1}\, b_n} \geqq b_1\, \alpha^{n-\frac{3}{2}},$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{r_n^{\rho}} < \left(\frac{\alpha^{\frac{3}{2}}}{b_1}\right)^{\rho} \sum_{n=1}^{\infty} \frac{1}{\alpha^{\rho n}}, \quad \alpha \geqq 4.$$

Since the right member is convergent, however small the positive number ϱ may be, it follows that *the order ϱ of $f(x)$ is zero.*

Formula (5), expressed in terms of the coefficient a_n, becomes

$$a_{n-1}^2 - 4\, a_n\, a_{n-2} \geqq 0.$$

From this we readily deduce

$$a_n \leqq \frac{a_0}{2^{n(n-1)}} \left(\frac{a_1}{a_0}\right)^n,$$

a formula that gives fairly precise information about the rate of growth of the maximum modulus of functions of the type included in Theorem *A*. This formula may advantageously replace the less simple formula of Petrovitch for all functions of the Petrovitch class having only positive coefficients, excepting the small residual class mentioned above in which not all of the b_n satisfy (5).

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NOTE ON AN AMBIGUOUS CASE OF APPROXIMATION*

BY

DUNHAM JACKSON

1. **Introduction.** Let $f(x)$ be a continuous function of period 2π. The writer has discussed to some extent the existence and properties of a trigonometric sum $T_{mn}(x)$, of order n at most, determined by the condition that the integral

$$\int_0^{2\pi} |f(x) - T_{mn}(x)|^m \, dx \tag{1}$$

shall have the smallest possible value, n being a given positive integer (or zero) and m a given number ≥ 1, not necessarily integral.† The purpose of the present note is to inquire what becomes of the problem for values of m less than 1.

For $m < 0$, it is clear that the value of (1) can be brought arbitrarily near to zero by taking a sufficiently large constant for $T_{mn}(x)$, but can not be made equal to zero, so that the problem has no significance. For $m = 0$, the integral (1) is always equal to 2π, unless there is a trigonometric sum which is identically equal to $f(x)$ throughout a point set of measure different from zero, in which case the integral is not defined. Attention may therefore be restricted to values of m between 0 and 1.

2. **Existence of an approximating function.** Let m be a number of the interval $0 < m < 1$. When m and n are given, the value of the integral (1) has a lower limit γ_{mn}, which is positive or zero. It is to be inquired first whether there exists a trigonometric sum $T_{mn}(x)$ for which the lower limit γ_{mn} is actually attained. The proof of existence given in the paper A for $m > 1$, and in B for $m = 1$, does not apply when $m < 1$. It is possible nevertheless

* Presented to the Society, December 30, 1920.

† See D. Jackson, *On functions of closest approximation*, these Transactions, vol. 22 (1921), pp. 117–128; *Note on a class of polynomials of approximation*, these Transactions, vol. 22 (1921), pp. 320–326; *On the convergence of certain trigonometric and polynomial approximations*, these Transactions, vol. 22 (1921), pp. 158–166.

These papers will be referred to by the letters A, B, C, respectively.

to give a proof which is valid for any value of $m > 0$, and which is as general in its scope as the earlier proofs mentioned, relating to a general class of approximating functions, of which trigonometric sums are only a very special case.

Let

$$p_1(x), p_2(x), \ldots, p_n(x)$$

be n functions of x, continuous and linearly independent in the interval $a \leqq x \leqq b$. Let

$$(2) \qquad \varphi(x) = c_1 p_1(x) + c_2 p_2(x) + \cdots + c_n p_n(x)$$

be an arbitrary linear combination of these functions with constant coefficients. The function $\varphi(x)$, then, might in particular be a trigonometric sum of order n, the interval being $(0, 2\pi)$; the number of terms would of course be $2n+1$, instead of n, but this is a mere matter of notation, and is of no consequence. Let H be the maximum of $|\varphi(x)|$ in (a, b).

We shall assume, without repeating the proof here, a lemma of Sibirani,* to the effect that there exists a constant P, completely determined by the set of functions $p_1(x), \ldots, p_n(x)$, such that

$$(3) \qquad |c_k| \leqq PH \qquad (k = 1, 2, \ldots, n),$$

for all functions $\varphi(x)$.

Because of the uniform continuity of the p's, there will exist a number $d > 0$, such that

$$(4) \qquad |p_k(x') - p_k(x'')| \leqq \frac{1}{2nP} \qquad (k = 1, 2, \ldots, n),$$

whenever

$$|x' - x''| \leqq d;$$

this d can be chosen once for all when the p's are given, since n and P are then determined, and it may be assumed that $d < \frac{1}{2}(b-a)$.

Let $\varphi(x)$ be any particular function of the form (2), and let x_0 be a point where $|\varphi|$ takes on its maximum value H. Then, if

$$(5) \qquad |x - x_0| \leqq d,$$

* For a simple proof (with slightly different notation) see the paper A, Lemma I.

it will follow from (3) and (4) that

$$|c_k p_k(x) - c_k p_k(x_0)| \leqq \frac{H}{2n} \qquad (k = 1, 2, \ldots, n),$$

$$|\varphi(x) - \varphi(x_0)| \leqq \frac{H}{2},$$

$$|\varphi(x)| \geqq \frac{H}{2}.$$

The last relation holds throughout an interval of length at least d, since the points of the interval (5) are in (a, b), on one side of x_0 at least, and hence, for $m > 0$,

$$G_m = \int_a^b |\varphi(x)|^m \, dx \geqq d\left(\frac{H}{2}\right)^m.$$

Consequently

$$H \leqq 2\left(\frac{G_m}{d}\right)^{\frac{1}{m}}.$$

$$|c_k| \leqq 2P\left(\frac{G_m}{d}\right)^{\frac{1}{m}} \qquad (k = 1, 2, \ldots, n).$$

That is,

There exists a constant Q_m, depending only on the p's and on the exponent m, such that

$$|c_k| \leqq Q_m G_m^{\frac{1}{m}} \qquad (k = 1, 2, \ldots, n),$$

for all functions $\varphi(x)$; this is true for any value of $m > 0$.

Let $f(x)$ be an arbitrary continuous function in (a, b), and let

$$g_m = \int_a^b |f(x) - \varphi(x)|^m \, dx.$$

When m, $f(x)$, and the p's are given, the value of g_m will have a lower limit, positive or zero, for all possible functions φ, and it is to be shown that there will exist at least one φ for which the lower limit is attained.

If $f(x)$ is linearly dependent on the p's, φ can be made identically equal to f, and g_m will thereby be made equal to its lower limit, zero.

If $f(x)$ is not linearly dependent on the p's, f and the p's together form a set of $n+1$ linearly independent functions. There will exist a constant q_m, depending only on f, the p's, and m, such that

$$|c_k| \leq q_m g_m^{\frac{1}{m}} \qquad (k = 1, 2, \ldots, n),$$

for all functions φ. So all the coefficients c_k in any φ which brings g_m near its lower limit will belong to a bounded region, which may be regarded as closed, in the space of n dimensions of which the c's are coördinates, and when g_m is regarded as a function of the c's in this closed region, its lower limit will be a minimum which is actually attained for a suitable choice of the c's. In any case, then,

There will be at least one function $\varphi(x)$ for which the value of g_m is a minimum.

3. **Non-uniqueness of the approximating function.** When $m > 1$, it can be shown that the minimizing function $\varphi(x)$ is unique.* The proof can be carried through also for $m = 1$, provided that the functions $p_k(x)$ are suitably specialized.† It is readily seen that the corresponding assertion is not true in general for $m < 1$, even if the discussion is restricted to functions $\varphi(x)$ which are finite trigonometric sums.

For example, let ϵ be a small positive quantity; let

$$f(x) = 1, \qquad \frac{\pi}{2} + \epsilon \leq x \leq \frac{3\pi}{2} - \epsilon;$$

$$f(x) = -1, \qquad 0 \leq x \leq \frac{\pi}{2} - \epsilon, \qquad \frac{3\pi}{2} + \epsilon \leq x \leq 2\pi;$$

and let $f(x)$ be linear and continuous from $(\pi/2) - \epsilon$ to $(\pi/2) + \epsilon$ and from $(3\pi/2) - \epsilon$ to $(3\pi/2) + \epsilon$, and of period 2π for all values of x. That is, as nearly as is consistent with continuity, $f(x)$ is equal to $+1$ throughout half a period, and to -1 throughout the other half. Let $\varphi(x)$ in this case be a finite trigonometric sum of order zero, that is, a constant, and let $m = \frac{1}{2}$. Then it is clear, from the symmetry of the definition of $f(x)$, that if g_m takes on its minimum value for $\varphi = c_0$, it will take on the same value for $\varphi = -c_0$, and so there will be two different minimizing constants, unless $c_0 = 0$; but the value of g_m for $\varphi = 0$ can be made arbitrarily near to 2π, and its value

* See A, § 6.

† See B, §§ 3, 4. The proof is written out for the polynomial case, but the method applies equally well to the problem of approximation by trigonometric sums; cf. B, end of § 1.

for $\varphi = \pm 1$ arbitrarily near to $\pi\sqrt{2}$, by taking ϵ sufficiently small, so that there are cases in which $\varphi = 0$ certainly does not give the minimum. For any other value of m between 0 and 1, the approximate values of g_m for $\varphi = 0$ and for $\varphi = \pm 1$ would be 2π and $2^m\pi$ respectively, and the conclusion would be the same. A similar example could be given for the case of polynomial approximation.

4. **Convergence.** In view of the failure of the property of uniqueness, it is perhaps all the more remarkable that the question of the convergence of the approximating functions toward the value $f(x)$, when m is held fast and n is allowed to become infinite, can be treated, in the special case of trigonometric approximation, in almost exactly the same way as for $m \geqq 1$. It is sufficient to refer to the discussion in the paper C, with the remark that the only property of the minimizing function used explicitly is that it gives the integral at least as small a value as a specified function of the same form. Let a trigonometric sum of order n or lower which gives the integral its minimum value, for given m and n, be referred to as *an approximating function* (no longer *the* approximating function) *of order n, corresponding to the exponent m*; then the reasoning of C will lead to the conclusion:

If $f(x)$ is a continuous function of period 2π which can be represented by a trigonometric sum of order n or lower with an error not exceeding ϵ_n, if $T_{mn}(x)$ is an approximating function of order n for $f(x)$, corresponding to the exponent m, and if ϱ_n is the maximum of $|f(x) - T_{mn}(x)|$, then

$$\varrho_n \leqq l_m n^{\frac{1}{m}} \epsilon_n,$$

where l_m ($= k_m + 1$ of the paper C) is a constant depending only on m.

The deduction of theorems on convergence follows as in C, on the basis of the general theorems on trigonometric approximation there referred to.* If there are two or more approximating functions for a given value of n, it will be immaterial for the truth of the conclusion which is chosen. There will be occasion now to assume that $f(x)$ has a derivative, of the first or of higher order, possessing a certain degree of continuity; the results need not be set down in further detail.

The discussion of convergence can be extended to values of $m < 1$ in the polynomial case as well.†

* See D. Jackson, *On the approximate representation of an indefinite integral*, etc., these Transactions, vol. 14 (1913), pp. 343 364.

† Cf. the paper C, §§ 5, 6.

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EXPANSIONS IN TERMS
OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

SECOND PAPER: MULTIPLE BIRKHOFF SERIES*

BY

CHESTER C. CAMP

1. SETTING OF THE PROBLEM

Consider the system of expressions

$$L_i(z) \equiv \frac{\partial^{n_i} z}{\partial x_i^{n_i}} + P_{i2}(x_i)\frac{\partial^{n_i-2} z}{\partial x_i^{n_i-2}} + \cdots + P_{in_i}(x_i)z \qquad (i=1,2,\ldots,\varkappa)$$

and their adjoints

$$M_i(z) \equiv (-1)^{n_i}\frac{\partial^{n_i} z}{\partial x_i^{n_i}} + (-1)^{n_i-2}\frac{\partial^{n_i-2}}{\partial x_i^{n_i-2}}[P_{i2}(x_i)z] + \cdots + P_{in_i}(x_i)z$$
$$(i=1,2,\ldots,\varkappa),$$

in which the coefficients $P_{ij}(x_i)$ $(i=1,2,\ldots,\varkappa;\ j=1,2,\ldots,n_i)$ are functions of the real variables x_i on closed intervals (a_i, b_i), which are continuous with their derivatives of all orders.

With the partial differential equation

$$(1)\qquad \sum_{i=1}^{\varkappa} L_i(u) + \lambda u = 0$$

and the boundary conditions

$$(2)\qquad T_{ij}(u) = 0 \qquad (i=1,2,\ldots,\varkappa;\ j=1,2,\ldots,n_i)$$

* Presented to the Chicago Section of the Society December 29, 1922. Acknowledgment is hereby made of the author's indebtedness to Professor R. D. Carmichael for suggesting the desirability of this extension as well as for hints concerning the manuscript.

we associate the adjoint equation

$$\sum_{i=1}^{\varkappa} M_i(v) + \lambda v = 0 \tag{3}$$

and certain adjoint boundary conditions

$$U_{ij}(v) = 0 \qquad (i = 1, 2, \ldots, \varkappa; j = 1, 2, \ldots, n_i); \tag{4}$$

where

$$\begin{aligned} T_{ij}(u) \equiv & \left[\alpha_{j0}^{(i)} u(x_1, x_2, \ldots, x_\varkappa) + \sum_{h=1}^{n_i-1} \alpha_{jh}^{(i)} \frac{\partial^h u}{\partial x_i^h}\right]_{x_i=a_i} \\ & + \left[\beta_{j0}^{(i)} u(x_1, x_2, \ldots, x_\varkappa) + \sum_{h=1}^{n_i-1} \beta_{jh}^{(i)} \frac{\partial^h u}{\partial x_i^h}\right]_{x_i=b_i} \end{aligned} \tag{5}$$

and $U_{ij}(v)$ is to be defined later.

If we make the substitutions

$$u \equiv \prod_{i=1}^{\varkappa} u_i(x_i), \qquad v \equiv \prod_{i=1}^{\varkappa} v_i(x_i), \tag{6}$$

and omit the trivial solutions, equation (1) reduces as in my first paper* to the system of ordinary differential equations

$$L_i(u_i) + \mu_i u_i = 0 \qquad (i = 1, 2, \ldots, \varkappa), \tag{7}$$

where $\sum \mu_i = \lambda$ and equation (3), by a suitable choice of μ's, to the adjoint system. Likewise the boundary conditions (2) upon being divided through by appropriate factors take the form

$$\begin{aligned} W_{ij}(u_i) \equiv & \alpha_{j0}^{(i)} u_i(a_i) + \beta_{j0}^{(i)} u_i(b_i) \\ & + \sum_{h=1}^{n_i-1} \left[\alpha_{jh}^{(i)} \frac{d^h u_i(a_i)}{dx_i^h} + \beta_{jh}^{(i)} \frac{d^h u_i(b_i)}{dx_i^h}\right] = 0 \\ & (i = 1, 2, \ldots, \varkappa; j = 1, 2, \ldots, n_i). \end{aligned} \tag{8}$$

* These Transactions, vol. 25 (1923), pp. 123–134.

For a particular i we then have a Birkhoff system. We shall restrict the α's and β's so that the boundary conditions are what he calls *regular*.* To each set W_{ij} corresponds one of his adjoint sets of boundary conditions,

$$V_{ij}(v_i) = 0 \qquad (i = 1, 2, \ldots, \varkappa;\ j = 1, 2, \ldots, n_i). \tag{9}$$

We now define $U_{ij}(v)$ to bear the same relation to $V_{ij}(v_i)$ that $T_{ij}(u)$ bears to $W_{ij}(u_i)$. Thus we ensure that for each i the parameter values $\mu_i^{(1)}, \mu_i^{(2)}, \ldots$ will be in general simple, and the principal solutions unique except for constant factors. For two distinct principal parameter values $\mu_i^{(m)}, \mu_i^{(n)}$ the orthogonal relation

$$\int_{a_i}^{b_i} u_i^{(m)}(x_i)\, v_i^{(n)}(x_i)\, dx_i = 0 \tag{10}$$

holds for the corresponding principal solutions.

2. The formal expansion

The object of this paper is to develop a more or less arbitrary function $f(x_1, x_2, \ldots, x_\varkappa)$ in the form

$$f = \sum_{h_1, h_2, \ldots, h_\varkappa = -\infty}^{\infty} \sum^{\infty} \cdots \sum^{\infty} C_{h_1, h_2, \ldots, h_\varkappa}\, U_{h_1, h_2, \ldots, h_\varkappa}(x_1, x_2, \ldots, x_\varkappa) \tag{11}$$

where $U_{h_1, h_2, \ldots, h_\varkappa} = \prod_{i=1}^{\varkappa} u_i^{(h_i)}(x_i)$ is the solution of (1), (2) corresponding to the characteristic values $\mu_i^{(h_i)}$ $(i = 1, 2, \ldots, \varkappa)$ for the systems (7), (8). By multiplying (11) by the solution of (3), (4) for the same values of μ_i, integrating and using (6), (10) we obtain in general

$$C_{h_1, h_2, \ldots, h_\varkappa} = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_\varkappa}^{b_\varkappa} f \prod_{i=1}^{\varkappa} v_i^{(h_i)}(x_i)\, dx_i}{\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_\varkappa}^{b_\varkappa} \prod_{i=1}^{\varkappa} u_i^{(h_i)}(x_i)\, v_i^{(h_i)}(x_i)\, dx_i}. \tag{12}$$

* These Transactions, vol. 9 (1908), pp. 382-3.

In case the parameter values are not all simple or the Green's functions for the systems (7), (8) have poles not all of the first order, we replace the corresponding term of (11) by

$$\int_{a_\varkappa}^{b_\varkappa}\int_{a_{\varkappa-1}}^{b_{\varkappa-1}}\cdots\int_{a_1}^{b_1} f(s_1, s_2, \ldots, s_\varkappa)\prod_{i=1}^{\varkappa} R_i^{(h_i)}(x_i, s_i)\, ds_i \tag{13}$$

where $R_i^{(h_i)}$ is the residue of the Green's function $G_i(x_i, s_i; \mu_i)$ $(i = 1, 2, \ldots, \varkappa)$ for the characteristic value $\mu_i^{(h_i)}$. Such an expansion may well be called a *multiple Birkhoff series.*

3. Convergence of the series

The value of the series is found as in my first paper* by taking the limit of contour integrals. The residue for a function of $\varkappa$ complex variables is defined as before. One makes the transformation $\mu_i = \varrho_i^{n_i}$ analogous to that of Birkhoff, $\lambda = \varrho^n$. Then one evaluates the limit of

$$\frac{1}{(2\pi\sqrt{-1})^\varkappa}\int_{\Gamma_\varkappa}\cdots\int_{\Gamma_1}\int_{a_\varkappa}^{b_\varkappa}\cdots\int_{a_1}^{b_1} f(s_1, \ldots, s_\varkappa)\prod_{i=1}^{\varkappa} G_i(x_i, s_i; \mu_i)\, ds_i\, d\mu_i$$

quite readily on account of the fact that the integrals separate in pairs. We may therefore state the

THEOREM. *Let $f(x_1, x_2, \ldots, x_\varkappa)$ be made up of a finite number of pieces in the region $a_i \leq x_i \leq b_i$ $(i = 1, 2, \ldots, \varkappa)$, each real, continuous, and possessing continuous partial derivatives. The multiple Birkhoff expansion connected with the partial differential equation* (1) *and the regular boundary conditions* (2) *converges to the mean value* $\frac{1}{2^\varkappa}\sum f(x_1 \pm 0, x_2 \pm 0, \ldots, x_\varkappa \pm 0)$ *at any interior point of the region. In any closed subregion in which f is continuous and possesses continuous partial derivatives the series converges uniformly to f.*

4. Applications to physical problems

Several of the most important differential equations of physics give rise to special cases of equation (1). For instance the wave equation $\partial^2\varphi/\partial t^2 = c\nabla^2\varphi$

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* Loc. cit.

has a large class of solutions of the form $\varphi = T(t)\, u(x, y, z)$. These depend on solutions of $T'' + \lambda t = 0$ and $c\nabla^2 u + \lambda u = 0$. The boundary conditions are usually regular.

The flow of heat, vibrations of a drumhead, and determination of potential also lead to forms of equation (1). Although the boundary conditions (2) for three dimensions seem to restrict us to values on a parallelepiped, it is important to notice that any transformation leading to generalized coördinates which changes an equation to another form included in (1) will allow us to treat more general boundary conditions involving values on pairs of mutually orthogonal surfaces. A similar remark applies to the system (7).

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CIRCULAR PLATES OF CONSTANT OR VARIABLE THICKNESS*

BY

CARL A. GARABEDIAN

CONTENTS

PART I. THE THEORY

1. Introduction. In a recent note† on *Circular plates of variable thickness*, Professor Birkhoff points the way to a new method in the theory of elasticity. Cauchy‡ treated the case of variable thickness, but like Poisson§ he assumed the displacements developable in ascending positive integral powers of the

* Presented to the American Mathematical Society, February 24, 1923. See also Comptes Rendus, vol. 177, No. 20, Nov. 12, 1923.

† Philosophical Magazine, ser. 6, No. 257, May, 1922.

‡ *Exercices de Mathématique*, vol. 3 (1828), pp. 277–284.

§ Memoir of 1828. Cf. also Todhunter and Pearson's *History*, vol. I.

distance from the middle-surface. Saint-Venant* and others were quick to observe that this assumption is quite untenable, and it appears that subsequent investigations in the direction taken by Poisson and Cauchy have as yet failed to reveal a suitable type of series development.† With reference to the question here at issue, it may be said that the object of the present paper is to present a powerful and direct method of series based upon the introduction of the "natural" parameter t employed by Birkhoff. The vital contribution of Birkhoff's note seems to be the notion of expansions in power series in t.

A first exposition is simplified by assuming the conditions such that the displacements take place in planes through an axis and are the same in all such planes. The advantage of this assumption is that the differential equations to be integrated are essentially *total*. Cylindrical coördinates are adopted, with the axis of z along the axis of the plate; and although the thickness is variable, it is assumed that there is symmetry in the "middle plane" $z = 0$.‡ The plate is taken to be homogeneous and isotropic, only slightly bent,§ and in general thin.

Let the equations of the upper and lower bases of the plate be respectively

$$z = a(r), \tag{1a}$$

$$z = -a(r). \tag{1b}$$

The method of series begins with the introduction of a parameter t by means of the relation

$$z = \zeta t. \tag{2}$$

At the same time we write

$$a(r) = \alpha(r)t. \tag{3}$$

* De Clebsch, *Théorie de l'Elasticité des Corps Solides*, 1883, annotated by Saint-Venant. Cf. also Love, *A Treatise on the Mathematical Theory of Elasticity*, third edition, 1920, p. 27.

† As this paper nears completion, I find a note by E. and M. Cosserat (Comptes Rendus, vol. 146, (1908), p. 169) which seems to be the most recent contribution to the subject in hand. Although this note recognizes the fact that the displacements are functions not only of the coördinates but also of certain geometric parameters, the writers deal only briefly with the question of series developments. They investigate the character of the displacements in the case of classic plate and rod problems, but in their method of attack there is no suggestion of series developments of the type on which we base the systematic treatment of the present paper. Messrs. Cosserat make no mention of plates of variable thickness or of rods of variable cross section.

‡ We shall exclude from the present paper any discussion of plates not symmetrical in $z = 0$.

§ Love, p. 462.

so that the equations of the bases may be written in the form

$$z = \alpha t, \tag{4a}$$

$$z = -\alpha t, \tag{4b}$$

or, when convenient, in the form

$$\zeta = \alpha, \tag{5a}$$

$$\zeta = -\alpha. \tag{5b}$$

Our *fundamental assumption* is that all quantities involved which are functions of r and z can be expanded in ascending powers of the parameter t: in particular, that

$$U(r, z) = U_0(r, \zeta) + U_1(r, \zeta)t + U_2(r, \zeta)t^2 + \cdots, \tag{6a}$$

$$w(r, z) = w_0(r, \zeta) + w_1(r, \zeta)t + w_2(r, \zeta)t^2 + \cdots, \tag{6b}$$

where U and w denote the radial and axial displacements respectively.

Let us write $a(r) = hf(r)$, $\alpha(r) = \eta f(r)$, where $h = \eta t$ is a constant multiplier. If $f(r) \equiv 1$, we have the special case of constant thickness: $a = h$, $\alpha = \eta$. With this notation, it turns out that the coefficient of t^n in (6a) or (6b) is a homogeneous polynomial† in ζ and η of degree n; that is, when the return is made to the original variable, the terms are homogeneous polynomials in z and h ordered according to degree. The coefficients are functions of r and the fixed radii of the plate, of f and its derivatives, and of the elastic constants and applied tractions. The manner in which these arguments enter is of no immediate concern, but it should be noted that the requirement of homogeneity in ζ and η places a very definite restriction upon the manner of occurrence of f and its derivatives. We shall find occasion in the sequel to make further comments with reference to the structure of our formulas.

The t plays a rôle analogous to that of the parameter in the definition of homogeneous function. Although a physical interpretation of transformation (2) is not immediate, there is suggested the picture of a "plate of reference" whose bases have the equations (5) and whose thickness is controlled by means of the parameter t. Or we may think of (5) as fixed, in which case the

† See the examples of Part II.

plate (4) has its thickness determined by t, and upon proper choice of t becomes the *given* plate.

The assumption of Cauchy and Poisson of developability in powers of z alone involves suppression of terms in the geometric parameter h. The introduction of the parameter t furnishes a method that incurs no loss of terms; and when the parameter t has served its purpose, it may be suppressed by setting $t = 1$ and replacing the Greek letters ζ and η by z and h, respectively.

We shall find it convenient to hold in abeyance the energy integral and calculus of variations method employed by Birkhoff, and proceed *directly* from the fundamental equations of elasticity. We take the point of view of prescribed pressures on the faces of the plate, and compute the displacements. This type of problem will require that the coefficients in the formulas of displacement be so determined that

(i) the *body forces* have the required values,
(ii) the *surface tractions* on the bases are as prescribed,
(iii) the *boundary conditions* at the edges are satisfied.

Our procedure is to be entirely in accord with the accepted elastic theory, as set forth, for example, by Love. Requirements (i), (ii), and (iii) are characterized by the usual equations, except that from our point of view they become identities in t. That which is new is (a) the *method* of proceeding from these equations to the actual solution of a problem, and (b) the applicability of the method to plates of *variable thickness*.

As a matter of notation, it is convenient to introduce an operator defined as follows†:

$$A^* = \frac{1}{r}\frac{\partial}{\partial r}(rA), \tag{7a}$$

$$= A' + \frac{A}{r}. \tag{7b}$$

† In the literature (Love, p. 202) one finds the Laplacian operators

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{8a}$$

$$\nabla_1^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}; \tag{8b}$$

when A is a function of r only, where $r^2 = x^2 + y^2$, it may be verified that

$$\nabla_1^2 A = A'^*, \tag{9a}$$

$$\nabla_1^4 A = A'^{*'*}. \tag{9b}$$

Here the subject of operation is in general a function of both r and ζ, and the accent denotes differentiation with regard to r. The functions that involve both r and ζ turn out to be polynomials in ζ, and consequently the operator and symbol of differentiation are attached ultimately to functions of r alone. Since the operator and the sign of differentiation occur frequently in alternation, we write $([(A^*)']^*)'$ in the more compact form: $A^{*\prime *\prime}$. For reference we note also the formulas

$$(AB)^* = A^* B + AB' \tag{10a}$$

$$= AB^* + A'B = (BA)^*. \tag{10b}$$

In integrating an equation of the type

$$A'^{*\prime *} = f(r), \tag{11a}$$

we make use of the form of definition (7 a), and write†

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dA}{dr}\right)\right\}\right] = f(r), \tag{11b}$$

from which A is readily found by a succession of quadratures. For example, the solution of

$$A^* = B \tag{11c}$$

is

$$A = \frac{1}{r}\int^r (rB)\,dr + \frac{C}{r}, \tag{11d}$$

C being a constant of integration.

Just as we speak of differentiating or integrating a function, it is useful to say that we "star" a function by multiplying by the independent variable, *differentiating*, and dividing by the independent variable; likewise, that we "anti-star" a function by multiplying by the independent variable, *integrating*, and dividing by the independent variable.

The object of the present paper is to develop enough of the formal machinery involved in our method to enable us to treat some important and interesting applications to plates meeting the requirements of symmetry already outlined. We shall consider first some cases of uniform thickness, in order both to check

† Clebsch, p. 735.

with the classical results and to gain familiarity with the details of the method from a formal viewpoint. Secondly, we shall give concrete examples sufficiently illustrative of the applications to plates of variable thickness.

What new light this program may shed upon the theory of circular plates of constant or variable thickness remains to be seen. We proceed at once to a systematic treatment along the lines indicated.

2. **The body force and surface traction conditions.** Using the star notation, the body force conditions† are given by

$$(12a)\qquad (\lambda+2\mu)U^{*\prime}+\frac{\lambda+\mu}{t}\frac{\partial w'}{\partial\zeta}+\frac{\mu}{t^2}\frac{\partial^2 U}{\partial\zeta^2}\underset{t}{\equiv}-\varrho F_r,$$

$$(12b)\qquad \frac{\lambda+2\mu}{t^2}\frac{\partial^2 w}{\partial\zeta^2}+\frac{\lambda+\mu}{t}\frac{\partial U^*}{\partial\zeta}+\mu w'^*\underset{t}{\equiv}-\varrho F_z.$$

Here the displacements U and w are given by (6). To simplify matters, we shall later take the body forces to be nil, but in general F_r and F_z are assumed developable in suitable power series in t of type (6). The relations (12) are identities in t, and the fact that coefficients of like powers of t are the same gives us our first hold on the coefficients that enter in (6).

To gain compactness in later formulas, it is desirable to introduce at once the elastic constants E and σ, denoting Young's modulus and Poisson's ratio respectively.‡ For reference we note that

$$(13a)\qquad E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu},\qquad \sigma=\frac{\lambda}{2(\lambda+\mu)};$$

$$(13b)\qquad \lambda=\frac{E\sigma}{(1+\sigma)(1-2\sigma)},\qquad \mu=\frac{E}{2(1+\sigma)}.$$

The identities (12) may now be written in the form§

$$(14a)\quad 2(1-\sigma)U^{*\prime}+\frac{1}{t}\frac{\partial w'}{\partial\zeta}+\frac{1-2\sigma}{t^2}\frac{\partial^2 U}{\partial\zeta^2}\underset{t}{\equiv}-\frac{2(1+\sigma)(1-2\sigma)\varrho F_r}{E},$$

$$(14b)\quad \frac{2(1-\sigma)}{t^2}\frac{\partial^2 w}{\partial\zeta^2}+\frac{1}{t}\frac{\partial U^*}{\partial\zeta}+(1-2\sigma)w'^*\underset{t}{\equiv}-\frac{2(1+\sigma)(1-2\sigma)\varrho F_z}{E}.$$

† Love, p. 141.

‡ Love, p. 101, § 69.

§ The equations (14) and (21) are divided by a constant factor as an aid to simplification of later formulas.

To derive the conditions for tractions applied to the faces of the plate, consider a tangent plane at an arbitrary point (r_0, θ_0, z_0) of the upper surface, $z = a = t\alpha$, and denote by ν the direction of the normal† taken positively upwards. We describe the traction across the tangent plane to the surface $r = r_0$ by means of its vector components $(\widehat{rr}, \widehat{\theta r}, \widehat{zr})$, with a similar notation for the tractions across the planes $\theta = \theta_0$ and $z = z_0$. The first letter shows the direction of the component traction, and the second letter indicates the plane across which it acts. The sense‡ is such that $\widehat{rr}$ is positive when it is a tension, negative when it is a pressure. When the components of the traction across the tangent plane are expressed in terms of the stress-components across the planes normal to r, θ, z, we have

$$\text{(15}a\text{)} \qquad \widehat{r\nu} = \widehat{rr}\cos(r, \nu) + \widehat{r\theta}\cos(\theta, \nu) + \widehat{rz}\cos(z, \nu),$$

$$\text{(15}b\text{)} \qquad \widehat{\theta\nu} = \widehat{\theta r}\cos(r, \nu) + \widehat{\theta\theta}\cos(\theta, \nu) + \widehat{\theta z}\cos(z, \nu),$$

$$\text{(15}c\text{)} \qquad \widehat{z\nu} = \widehat{zr}\cos(r, \nu) + \widehat{z\theta}\cos(\theta, \nu) + \widehat{zz}\cos(z, \nu).$$

In cylindrical coördinates, the stress-components are given in terms of the strain-components and the displacements U and w as follows:§

$$\text{(16}a\text{)} \qquad \widehat{rr} = \lambda\Delta + 2\mu e_{rr} = \lambda\Delta + 2\mu U',$$

$$\text{(16}b\text{)} \qquad \widehat{\theta\theta} = \lambda\Delta + 2\mu e_{\theta\theta} = \lambda\Delta + 2\mu\frac{U}{r},$$

$$\text{(16}c\text{)} \qquad \widehat{zz} = \lambda\Delta + 2\mu e_{zz} = \lambda\Delta + 2\mu\frac{1}{t}\,\frac{\partial w}{\partial \zeta},$$

$$\text{(16}d\text{)} \qquad \widehat{\theta z} = \widehat{z\theta} = \mu e_{\theta z} = 0,$$

$$\text{(16}e\text{)} \qquad \widehat{zr} = \widehat{rz} = \mu e_{zr} = \mu\left(\frac{1}{t}\,\frac{\partial U}{\partial \zeta} + w'\right),$$

$$\text{(16}f\text{)} \qquad \widehat{r\theta} = \widehat{\theta r} = \mu e_{r\theta} = 0.$$

† Love, p. 76.
‡ Love, p. 76. Cf. Love, p. 461, and also § 5 of this paper.
§ Love, p. 101, 141.

where

$$\Delta = U^* + \frac{1}{t}\frac{\partial w}{\partial \zeta},$$

and U and w are given by (6).

Since the slope of the tangent to $z = t\alpha$ is $t\alpha'$, the desired direction cosines in (15) are

$$\text{(17)}\quad \cos(r,\nu) = \frac{-t\alpha'}{\sqrt{1+t^2\alpha'^2}},\quad \cos(\theta,\nu) = 0,\quad \cos(z,\nu) = \frac{1}{\sqrt{1+t^2\alpha'^2}}.$$

Let the applied tractions normal to the upper and lower bases be S^ν and S_ν respectively, where

$$\text{(18}a\text{)}\qquad S^\nu = S^0 + S^1 t + S^2 t^2 + \cdots,$$

$$\text{(18}b\text{)}\qquad S_\nu = S_0 + S_1 t + S_2 t^2 + \cdots,$$

the coefficients in these developments being functions of r only. By virtue of (17) the radial and axial components may be written

$$\text{(19}a\text{)}\qquad R^\nu = \frac{-t\alpha' S^\nu}{\sqrt{1+t^2\alpha'^2}},\qquad Z^\nu = \frac{S^\nu}{\sqrt{1+t^2\alpha'^2}}.$$

$$\text{(19}b\text{)}\qquad R_\nu = \frac{t\alpha' S_\nu}{\sqrt{1+t^2\alpha'^2}},\qquad Z_\nu = \frac{S_\nu}{\sqrt{1+t^2\alpha'^2}}.$$

If the plate is of constant thickness, $\alpha' = 0$ and $R^\nu \equiv R_\nu \equiv 0$, $Z^\nu \equiv S^\nu$, $Z_\nu \equiv S_\nu$.

We thus find for the surface tractions on the bases the identities

$$\text{(20}a\text{)}\qquad \widehat{r\nu}\Big|_{z=a} = \frac{-t\alpha'\widehat{rr} + \widehat{zr}}{\sqrt{1+t^2\alpha'^2}}\Big|_{\zeta=a} \underset{t}{\equiv} \frac{-t\alpha' S^\nu}{\sqrt{1+t^2\alpha'^2}} = R^\nu,$$

$$\text{(20}b\text{)}\qquad \widehat{r\nu}\Big|_{z=-a} = \frac{t\alpha'\widehat{rr} + \widehat{zr}}{\sqrt{1+t^2\alpha'^2}}\Big|_{\zeta=-a} \underset{t}{\equiv} \frac{t\alpha' S_\nu}{\sqrt{1+t^2\alpha'^2}} = R_\nu,$$

$$\text{(20}c\text{)}\qquad \widehat{z\nu}\Big|_{z=a} = \frac{-t\alpha'\widehat{zr} + \widehat{zz}}{\sqrt{1+t^2\alpha'^2}}\Big|_{\zeta=a} \underset{t}{\equiv} \frac{S^\nu}{\sqrt{1+t^2\alpha'^2}} = Z^\nu,$$

$$\text{(20}d\text{)}\qquad \widehat{z\nu}\Big|_{z=-a} = \frac{t\alpha'\widehat{zr} + \widehat{zz}}{\sqrt{1+t^2\alpha'^2}}\Big|_{\zeta=-a} \underset{t}{\equiv} \frac{S_\nu}{\sqrt{1+t^2\alpha'^2}} = Z_\nu.$$

Recall that the sense is such that the applied tractions are tensions when positive, pressures when negative.

Using formulas (16) and the elastic constants E and σ, we have as final forms for the surface traction conditions the following:

$$\begin{aligned} -2\alpha'\left\{\sigma\left(U^*t+\frac{\partial w}{\partial\zeta}\right)+(1-2\sigma)U't\right\} \\ +(1-2\sigma)\left\{\frac{1}{t}\frac{\partial U}{\partial\zeta}+w'\right\}\Big|_{\zeta=\alpha} \underset{t}{\equiv} \frac{-2(1+\sigma)(1-2\sigma)t\alpha'\mathrm{S}^{\nu}}{E}, \end{aligned} \tag{21a}$$

$$\begin{aligned} 2\alpha'\left\{\sigma\left(U^*t+\frac{\partial w}{\partial\zeta}\right)+(1-2\sigma)U't\right\} \\ +(1-2\sigma)\left\{\frac{1}{t}\frac{\partial U}{\partial\zeta}+w'\right\}\Big|_{\zeta=-\alpha} \underset{t}{\equiv} \frac{2(1+\sigma)(1-2\sigma)t\alpha'\mathrm{S}_{\nu}}{E}, \end{aligned} \tag{21b}$$

$$\begin{aligned} -\alpha'(1-2\sigma)\left\{\frac{\partial U}{\partial\zeta}+w't\right\}+2\sigma U^*+2(1-\sigma)\frac{1}{t}\frac{\partial w}{\partial\zeta}\Big|_{\zeta=\alpha} \\ \underset{t}{\equiv} \frac{2(1+\sigma)(1-2\sigma)\mathrm{S}^{\nu}}{E}, \end{aligned} \tag{21c}$$

$$\begin{aligned} \alpha'(1-2\sigma)\left\{\frac{\partial U}{\partial\zeta}+w't\right\}+2\sigma U^*+2(1-\sigma)\frac{1}{t}\frac{\partial w}{\partial\zeta}\Big|_{\zeta=-\alpha} \\ \underset{t}{\equiv} \frac{2(1+\sigma)(1-2\sigma)\mathrm{S}_{\nu}}{E}. \end{aligned} \tag{21d}$$

3. Determination of formulas of displacement that satisfy the body force conditions. The body force conditions (14) will determine the coefficients in the formulas of displacement up to arbitrary functions of r. Next the traction conditions (21) come into play to furnish differential equations for the determination of these functions of r; and finally the constants of integration are fixed by means of the boundary conditions at the edges.

To handle expeditiously the program just outlined, it is desirable to introduce a notation. Let the equations obtained from ($14a$) and ($14b$) by equating coefficients of t^n be denoted by $\mathrm{F}_r^{(n)}=0$, $\mathrm{F}_z^{(n)}=0$, respectively. Thus the body force conditions yield the equations

$$\mathrm{F}_r^{(n)}=0,\ \mathrm{F}_z^{(n)}=0 \qquad (n=-2,-1,0,1,2,\ldots). \tag{22}$$

By an obvious extension of this notation, the traction conditions (21 a), (21 b), (21 c), (21 d) may be said to yield respectively the equations

$$(23)\qquad R^{(n)}=0,\ R_{(n)}=0,\ Z^{(n)}=0,\ Z_{(n)}=0\qquad (n=-1,0,1,2,\dots).$$

We are now in a position to proceed to the determination of the coefficients in (6). The present paragraph is devoted to finding formulas of displacement that satisfy the identities (14) under the assumption that the body forces F_r, F_z vanish identically.

From the equations $F_r^{(-2)}=0$, $F_z^{(-2)}=0$ we find

$$(24)\qquad \frac{\partial^2 U_0}{\partial\zeta^2}=0,\qquad \frac{\partial^2 w_0}{\partial\zeta^2}=0.$$

Integrating, we have

$$(25a)\qquad U_0=U_{0b}+U_{0a}\zeta,$$

$$(25b)\qquad w_0=w_{0b}+w_{0a}\zeta,$$

where U_{0a}, U_{0b}, w_{0a}, w_{0b} are functions of r only, and are subsequently to be determined by means of the surface traction conditions.

In general, the equations $F_r^{(n-2)}=0$, $F_z^{(n-2)}=0$ determine U_n, w_n as polynomials in ζ, with coefficients that are functions of the U_a's, U_b's, w_a's and w_b's. Integration will introduce two new arbitrary functions of r, and the coefficients of all powers of ζ higher than the first will be known in terms of functions of r previously introduced; for example:

$$(26a)\qquad U_n=U_{nb}+U_{na}\zeta+\text{(terms of higher order in }\zeta),$$

$$(26b)\qquad w_n=w_{nb}+w_{na}\zeta+\text{(terms of higher order in }\zeta).$$

Finally, we note that the computation of U_n will require that w_{n-1} and U_{n-2} should already have been computed, and that the computation of w_n will require the previous computation of U_{n-1} and w_{n-2}.

Before proceeding to (22), $n=-1$, an important simplification of (25) results if we first introduce the traction conditions (23), $n=-1$. These relations are found to be

$$(27a)\qquad R^{(-1)} = \frac{\partial U_0}{\partial \zeta}\bigg|_{\zeta=a} = 0, \qquad R_{(-1)} = \frac{\partial U_0}{\partial \zeta}\bigg|_{\zeta=-a} = 0,$$

$$(27b)\qquad Z^{(-1)} = \frac{\partial w_0}{\partial \zeta}\bigg|_{\zeta=a} = 0, \qquad Z_{(-1)} = \frac{\partial w_0}{\partial \zeta}\bigg|_{\zeta=-a} = 0,$$

and they require that $U_{0a} \equiv w_{0a} \equiv 0$; in other words, the leading terms in the formulas of displacement are independent of ζ.

From $F_r^{(-1)} = 0$, $F_z^{(-1)} = 0$, we have

$$(28a)\qquad \frac{\partial^2 U_1}{\partial \zeta^2} = -\frac{1}{1-2\sigma}\frac{\partial w_0'}{\partial \zeta},$$

$$(28b)\qquad \frac{\partial^2 w_1}{\partial \zeta^2} = -\frac{1}{2(1-\sigma)}\frac{\partial U_0^*}{\partial \zeta};$$

but the right-hand members vanish, since $U_0 = U_{0b}$ and $w_0 = w_{0b}$ are functions of r only, and we find

$$(29a)\qquad U_1 = U_{1b} + U_{1a}\zeta,$$

$$(29b)\qquad w_1 = w_{1b} + w_{1a}\zeta.$$

Before proceeding to the computation of U_2 and w_2 it will simplify matters to consider the traction conditions (23) for $n=0$. To deal conveniently with these and subsequent traction equations, we introduce a further special notation. *Add* equations (21 a) and (21 b), and denote by $R^n = 0$ (without parentheses enclosing the superscript) the equation obtained by equating coefficients of t^n in the resulting equation; similarly, let $R_n = 0$ denote the equation obtained by equating coefficients of t^n in the equation which results when (21 b) is *subtracted* from (21 a). Finally, let $Z^n = 0$ and $Z_n = 0$ denote analogous equations derived from the sum and difference, respectively, of (21 c) and (21 d). The surface traction differential equations (23) become

$$(30)\qquad R^n = 0,\quad R_n = 0,\quad Z^n = 0,\quad Z_n = 0 \quad (n = -1, 0, 1, 2, \cdots).$$

We have already disposed of the case $n = -1$; and since $R^{(0)} = 0$, $R_{(0)} = 0$ require that

$$(31)\qquad \frac{\partial U_1}{\partial \zeta} + w_0'\bigg|_{\zeta=\pm a} = U_{1a} + w_{0b}' = 0,$$

we have directly

$$U_{1a} = -w'_{0b}, \tag{31b}$$

and there is no advantage to be gained by computing $R^0 = 0$, $R_0 = 0$. But for $Z^{(0)} = 0$, $Z_{(0)} = 0$, we find

$$w_{1a} = -\frac{\sigma}{1-\sigma} U^*_{0b} + \frac{(1+\sigma)(1-2\sigma)S^0}{(1-\sigma)E}, \tag{32a}$$

$$w_{1a} = -\frac{\sigma}{1-\sigma} U^*_{0b} + \frac{(1+\sigma)(1-2\sigma)S_0}{(1-\sigma)E}; \tag{32b}$$

and these results are more concisely given by $Z_0 = 0$ and $Z^0 = 0$, namely

$$S^0 \equiv S_0, \tag{33a}$$

$$w_{1a} = -\frac{\sigma}{1-\sigma} U^*_{0b} + \frac{(1+\sigma)(1-2\sigma)S^0_0}{(1-\sigma)E}, \tag{33b}$$

where $S^0 \equiv S_0 \equiv S^0_0$. We note that a tension or pressure to be of as low order as t^0 must be the *same* on both bases. Indeed, we shall find that the difference between the applied tractions on the two bases is of order t^3, or else nil.

We turn next to $F^{(0)}_r = 0$, $F^{(0)}_z = 0$, and find

$$U_2 = U_{2b} + U_{2a}\zeta - \frac{1}{2(1-2\sigma)} w'_{1a}\zeta^2 - \frac{1-\sigma}{1-2\sigma} U^{*\prime}_{0b}\zeta^2. \tag{34a}$$

$$w_2 = w_{2b} + w_{2a}\zeta - \frac{1}{4(1-\sigma)} U^*_{1a}\zeta^2 - \frac{1-2\sigma}{4(1-\sigma)} w'^*_{0b}\zeta^2. \tag{34b}$$

To simplify these results, we have at our disposal (31*b*) and (33*b*). If we further restrict S^0_0 (in general a function of r) to be a constant,† it follows from (33*b*) that

$$w'_{1a} = -\frac{\sigma}{1-\sigma} U^{*\prime}_{0b}, \tag{35}$$

† It should perhaps be remarked at this point that the restrictions imposed in the present paragraph are not such as to obscure the method of procedure in the general case. Our aim is not so much to be exhaustive in treatment as to cover adequately a number of sufficiently suggestive cases.

and we may write (34) as follows:

$$(36a) \qquad U_2 = U_{2b} + U_{2a}\,\zeta - \frac{2-\sigma}{2(1-\sigma)}\,U^{*\prime}_{0b}\,\zeta^2.$$

$$(36b) \qquad w_2 = w_{2b} + w_{2a}\,\zeta + \frac{\sigma}{2(1-\sigma)}\,w_0^{\prime *}\,\zeta^2.$$

Turning again to the traction conditions, we find† for $R^1 = 0$, $R_1 = 0$

$$(37a) \qquad w'_{1b} + U_{2a} = 0,$$

$$(37b) \qquad U^{*\prime}_{0b}\,\alpha + \left[U'_{0b} + \sigma\frac{U_{0b}}{r}\right]\alpha' = \frac{(1+\sigma)(1-2\sigma)S_0^0\,\alpha'}{E}.$$

By means of (37a) we eliminate U_{2a} once for all; (37b) is the first of a system of differential equations which we shall study separately in the following paragraph.

For $Z_1 = 0$, $Z^1 = 0$ the results are, respectively,

$$(38a) \qquad S^1 = S_1,$$

$$(38b) \qquad w_{2a} = -\frac{\sigma}{1-\sigma}\,U^*_{1b} + \frac{(1+\sigma)(1-2\sigma)S_1^1}{(1-\sigma)E},$$

where $S^1 = S_1 = S_1^1$.

We here introduce a further restriction,‡ and set $S_1^1 = 0$. Indeed, in prescribing the load we propose to consider only three cases of tractions applied to the faces of the plate, namely

$$(\text{i}) \qquad S^\nu = S_\nu = 0,$$

$$(\text{ii}) \qquad S^\nu = S_\nu = S_0^0,$$

$$(\text{iii}) \qquad S^\nu = S^3\,t^3,\ S_\nu = 0.$$

† We use (33), and the fact that $S_0^{0'} = 0$.

‡ This is no restriction physically, for we are prescribing the load.

This means that all the S's, that is, coefficients in (18), vanish identically save S_0^0 and S^3, and it will simplify the presentation to have these facts before us at the outset.

If $S_1^1 \equiv 0$, we have, from (38*b*),

$$w_{2a} = -\frac{\sigma}{1-\sigma} U_{1b}^{*}; \tag{39}$$

and recalling (37*a*), it is clear that both U_{2a} and w_{2a} may now be eliminated once for all. Bearing this fact in mind, and making use of (35), we have for $F_r^{(1)} = 0$, $F_z^{(1)} = 0$ the following simplified forms:

$$U_3 = U_{3b} + U_{3a}\zeta - \frac{2-\sigma}{2(1-\sigma)} U_{1b}^{*\prime}\zeta^2 + \frac{2-\sigma}{6(1-\sigma)} w_{0b}^{\prime *\prime}\zeta^3, \tag{40a}$$

$$w_3 = w_{3b} + w_{3a}\zeta + \frac{\sigma}{2(1-\sigma)} w_{1b}^{\prime *}\zeta^2 + \frac{1+\sigma}{6(1-\sigma)} U_{0b}^{*\prime *}\zeta^3. \tag{40b}$$

Finally we note also that $Z_2 = 0$ gives $S^2 \equiv S_2$, using (37*a*), and we set $S^2 \equiv S_2 \equiv S_2^2 \equiv 0$.

In this paragraph we consider no further traction equations; indeed, the later differential equations become rather involved, and we devote a separate paragraph to their systematic treatment. We have attained our object; namely, to obtain in the formulas of displacement such simplifications as are possible without restriction upon the applications. Aside, then, from the traction conditions already employed, we confine ourselves to body force equations only, and thus obtain general forms for U and w that satisfy (14) formally. These play so important a rôle that we record them below through the terms in t^6:

$$\begin{aligned} U = {} & U_{0b} + [U_{1b} - w_{0b}'\zeta]\, t + \left[U_{2b} - w_{1b}'\zeta - \frac{2-\sigma}{2(1-\sigma)} U_{0b}^{*\prime}\zeta^2\right] t^2 \\ & + \left[U_{3b} + U_{3a}\zeta - \frac{2-\sigma}{2(1-\sigma)} U_{1b}^{*\prime}\zeta^2 + \frac{2-\sigma}{6(1-\sigma)} w_{0b}^{\prime *\prime}\zeta^3\right] t^3 + \Big[U_{4b} + U_{4a}\zeta \\ & - \frac{1}{2(1-2\sigma)} w_{3a}'\zeta^2 - \frac{1-\sigma}{1-2\sigma} U_{\cdot b}^{*\prime}\zeta^2 + \frac{2-\sigma}{6(1-\sigma)} w_{1b}^{\prime *\prime}\zeta^3 \\ & + \frac{3-\sigma}{24(1-\sigma)} U_{0b}^{*\prime *\prime}\zeta^4\Big] t^4 + \Big[U_{5b} + U_{5a}\zeta - \frac{1}{2(1-2\sigma)} w_{4a}\zeta^2 \end{aligned} \tag{41a}$$

$$-\frac{1-\sigma}{1-2\sigma}U_{3b}^{*\prime}\zeta^2-\frac{3-2\sigma}{12(1-\sigma)}U_{3a}^{*\prime}\zeta^3+\frac{1}{12(1-\sigma)}w_{2b}^{\prime*\prime}\zeta^3$$

$$+\frac{3-\sigma}{24(1-\sigma)}U_{1b}^{*\prime*\prime}\zeta^4-\frac{3-\sigma}{120(1-\sigma)}w_{0b}^{\prime*\prime*\prime}\zeta^5\Big]t^5$$

$$+\Big[U_{6b}+U_{6a}\zeta-\frac{1}{2(1-2\sigma)}w_{5a}^{\prime}\zeta^2-\frac{1-\sigma}{1-2\sigma}U_{4b}^{*\prime}\zeta^2-\frac{3-2\sigma}{12(1-\sigma)}U_{4a}^{*\prime}\zeta^3$$

$$+\frac{1}{12(1-\sigma)}w_{3b}^{\prime*\prime}\zeta^3+\frac{1}{12(1-2\sigma)}w_{3a}^{\prime*\prime}\zeta^4+\frac{3-2\sigma}{24(1-2\sigma)}U_{2b}^{*\prime*\prime}\zeta^4$$

$$-\frac{3-\sigma}{120(1-\sigma)}w_{1b}^{\prime*\prime*\prime}\zeta^5-\frac{4-\sigma}{720(1-\sigma)}U_{0b}^{*\prime*\prime*\prime}\zeta^6\Big]t^6+\cdots;$$

$$(41\,b)\qquad w=w_{0b}+[w_{1b}+w_{1a}\zeta]\,t+\Big[w_{2b}-\frac{\sigma}{1-\sigma}U_{1b}^{*}\zeta+\frac{\sigma}{2(1-\sigma)}w_{0b}^{\prime*}\zeta^2\Big]t^2$$

$$+\Big[w_{3b}+w_{3a}\zeta+\frac{\sigma}{2(1-\sigma)}w_{1b}^{\prime*}\zeta^2+\frac{1+\sigma}{6(1-\sigma)}U_{0b}^{*\prime*}\zeta^3\Big]t^3+\Big[w_{4b}+w_{4a}\zeta$$

$$-\frac{1}{4(1-\sigma)}U_{3a}^{*}\zeta^2-\frac{1-2\sigma}{4(1-\sigma)}w_{2b}^{\prime*}\zeta^2+\frac{1+\sigma}{6(1-\sigma)}U_{1b}^{*\prime*}\zeta^3$$

$$-\frac{1+\sigma}{24(1-\sigma)}w_{0b}^{\prime*\prime*}\zeta^4\Big]t^4+\Big[w_{5b}+w_{5a}\zeta-\frac{1}{4(1-\sigma)}U_{4a}^{*}\zeta^2$$

$$-\frac{1-2\sigma}{4(1-\sigma)}w_{3b}^{\prime*}\zeta^2+\frac{\sigma}{3(1-2\sigma)}w_{3a}^{\prime*}\zeta^3+\frac{1}{6(1-2\sigma)}U_{2b}^{*\prime*}\zeta^3$$

$$-\frac{1+\sigma}{24(1-\sigma)}w_{1b}^{\prime*\prime*}\zeta^4-\frac{2+\sigma}{120(1-\sigma)}U_{0b}^{*\prime*\prime*}\zeta^5\Big]t^5+\Big[w_{6b}+w_{6a}\zeta$$

$$-\frac{1}{4(1-\sigma)}U_{5a}^{*}\zeta^2-\frac{1-2\sigma}{4(1-\sigma)}w_{4b}^{\prime*}\zeta^2+\frac{\sigma}{3(1-2\sigma)}w_{4a}^{\prime*}\zeta^3$$

$$+\frac{1}{6(1-2\sigma)}U_{3b}^{*\prime*}\zeta^3+\frac{1}{24(1-\sigma)}U_{3a}^{*\prime*}\zeta^4-\frac{\sigma}{24(1-\sigma)}w_{2b}^{\prime*\prime*}\zeta^4$$

$$-\frac{2+\sigma}{120(1-\sigma)}U_{1b}^{*\prime*\prime*}\zeta^5+\frac{2+\sigma}{720(1-\sigma)}w_{0b}^{\prime*\prime*\prime*}\zeta^6\Big]t^6+\cdots.$$

4. **Four systems of differential equations obtained from the surface traction conditions.** In the preceding paragraph, the surface traction equations (30) were disposed of for the cases $n = -1$ and $n = 0$; we made use also of the equations $R^1 = 0$, $Z^1 = 0$, $Z_1 = 0$, $Z_2 = 0$. The remaining differential equations are considered in the present paragraph.

The differential equations (30) are total, r being the independent variable. Furthermore, the equations are linear, with coefficients that are functions of α, and for plates of uniform thickness these coefficients are obviously constants. The functions of r that are arbitrary in (41) are determined by these differential equations up to the constants of integration.

When the surface traction differential equations are written out in full, it is found that they fall naturally into four systems according as they involve the unknowns U_{0b}, U_{1b}, w_{0b}, w_{1b}. The facts with reference to the equations and unknowns which enter in these four systems are given in the following table:

System	*Equations*	*Functions* $n = 1, 2, \ldots, m.$	*Number of equations (or functions)* $m = 0, 1, 2, \ldots$
U_{0b}	$R_{2n+1} = 0, \quad n = 0, 1, 2, \ldots, m;$ $Z^{2n} = 0, \quad n = 1, 2, \ldots, m.$	$U_{0b};$ $U_{2n,b};\ w_{2n+1,a}.$	$2m+1$
U_{1b}	$R_{2n+2} = 0, \quad n = 0, 1, 2, \ldots, m;$ $Z^{2n+1} = 0, \quad n = 1, 2, \ldots, m.$	$U_{1b};$ $U_{2n+1,b};\ w_{2n+2,a}.$	$2m+1$
w_{0b}	$R^{2n+2} = 0,$ $Z_{2n+3} = 0, \quad n = 0, 1, 2, \ldots, m.$	$w_{0b};\ w'_{2m+2,b} + U_{2m+3,a};$ $w_{2n,b};\ U_{2n+1,a}.$	$2m+2$
w_{1b}	$R^{2n+3} = 0,$ $Z_{2n+4} = 0, \quad n = 0, 1, 2, \ldots, m.$	$w_{1b};\ w'_{2m+3,b} + U_{2m+4,a};$ $w_{2n+1,b};\ U_{2n+2,a}.$	$2m+2$

The meaning of the table is best understood after a few of the differential equations of each system have been written out in full. For this purpose, and for the more important applications as well, it will suffice to consider only $m = 0, 1$.

(*a*) THE U_{0b} SYSTEM

We have already met the first differential equation of the U_{0b} system. It is $R_1 = 0$, an equation which we found it convenient to compute while we were considering $R^1 = 0$. We begin, then, with (37*b*):

$$U_{0b}^{*\prime}\,\alpha + \left[U_{0b}' + \sigma\frac{U_{0b}}{r}\right]\alpha' = \frac{(1+\sigma)(1-2\sigma)S_0^0\,\alpha'}{E}. \tag{42}$$

For $m=1$, we have to compute the equations $Z^2=0$, $R_3=0$. Since $S_2^2 \equiv 0$, we find, for $Z^2=0$,

$$\frac{1-2\sigma}{1-\sigma}(U_{0b}^{*\prime}\,\alpha^2)^* + 2\sigma U_{2b}^* + 2(1-\sigma)\,w_{3a} = 0. \tag{43a}$$

where, by formula (10*a*),

$$(U_{0b}^{*\prime}\,\alpha^2)^* = U_{0b}^{*\prime *}\,\alpha^2 + 2\,U_{0b}^{*\prime}\,\alpha\alpha'. \tag{43b}$$

The equation $R_3 = 0$, when written out in full, becomes

$$\begin{aligned}\frac{2(1-2\sigma)}{3(1-\sigma)}U_{0b}^{*\prime *\prime}\,\alpha^3 &+ \frac{\sigma(1-2\sigma)}{1-\sigma}U_{0b}^{*\prime *}\,\alpha^2\alpha' + \frac{(2-\sigma)(1-2\sigma)}{1-\sigma}U_{0b}^{*\prime\prime}\,\alpha^2\alpha\\ &-2(1-\sigma)U_{2b}^{*\prime}\,\alpha - 2\sigma(U_{2b}^* + w_{3a})\alpha' - 2(1-2\sigma)U_{2b}'\,\alpha'\\ &-2\sigma w_{3a}'\,\alpha = 0.\end{aligned} \tag{44}$$

The equations (43*a*) and (44) determine U_{2b} and w_{3a}, since the function U_{0b} is known by (42). Without writing down the next two equations of the system, it is clear that they will involve the three functions already obtained and two new functions, U_{4b} and w_{5a}, which they will determine.

(*b*) THE U_{1b} SYSTEM

Although the differential equations of this system may be readily obtained from those of the U_{0b} system by merely advancing all the indices by 1, their importance in the applications to follow makes it desirable that they be explicitly set down at this point. (Also, the restrictions we have imposed upon the S's are going to alter the appearance of our equations somewhat, in

particular the first three, so that the rule which advances indices is not easily applied.)

The first equation ($m = 0$) is $R_2 = 0$. Since $S_1^1 \equiv 0$, we find

$$U_{1b}^{*\prime} \alpha + \left[U'_{1b} + \sigma \frac{U_{1b}}{r} \right] \alpha' = 0. \tag{45}$$

When $m = 1$, the equations are $Z^3 = 0$ and $R_4 = 0$; or

$$\frac{1-2\sigma}{1-\sigma} (U_{1b}^{*\prime} \alpha^2)^* + 2\sigma U_{3b}^* + 2(1-\sigma) w_{4a} = \frac{(1+\sigma)(1-2\sigma)(S^3 + S_3)}{E}, \tag{46}$$

and

$$\begin{aligned} &\frac{2(1-2\sigma)}{3(1-\sigma)} U_{1b}^{*\prime *\prime} \alpha^3 + \frac{\sigma(1-2\sigma)}{1-\sigma} U_{1b}^{*\prime *} \alpha^2 \alpha' + \frac{(2-\sigma)(1-2\sigma)}{1-\sigma} U_{1b}^{*\prime\prime} \alpha^2 \alpha' \\ &-2(1-\sigma) U_{3b}^{*\prime} \alpha - 2\sigma(U_{3b}^* + w_{4a}) \alpha' - 2(1-2\sigma) U'_{3b} \alpha' - 2\sigma w'_{4a} \alpha \\ &= \frac{(1+\sigma)(1-2\sigma)(S^3 + S_3)\alpha'}{E}. \end{aligned} \tag{47}$$

The last two equations determine U_{3b} and w_{4a}, since (45) has determined U_{1b}. S^3 and S_3 are in general distinct. We shall take $S_3 \equiv 0$ in agreement with § 3. In that paragraph we pointed out a fact which is now established; namely, that if S^ν and S_ν are distinct, their difference is of order t^3.

We note that the U_{0b} and U_{1b} systems are formally the same; if, in particular, the surface tractions on the bases are nil, the corresponding unknowns of the two systems differ only in the constants of integration.

(c) THE w_{0b} SYSTEM

For $m = 0$, the table gives two differential equations of the w_{0b} system; namely, $R^2 = 0$ and $Z_3 = 0$. We have, in full,

$$\frac{1}{1-\sigma} w_{0b}^{\prime *\prime} \alpha^2 + \frac{2}{1-\sigma} \left[w''_{0b} + \sigma \frac{w'_{0b}}{r} \right] \alpha\alpha' + (w'_{2b} + U_{3a}) = 0, \tag{48}$$

$$\left[\frac{1}{3(1-\sigma)} w_{0b}^{\prime *\prime} \alpha^3 + (w'_{2b} + U_{3a}) \alpha \right]^* = -\frac{(1+\sigma)(S^3 - S_3)}{E}. \tag{49}$$

These equations determine w_{0b} and $w'_{2b} + U_{3a}$, if we admit as a single unknown the combination $w'_{2b} + U_{3a}$. It will be found that the next two equations, $m = 1$, determine w_{2b} and U_{3a} individually. Since we are interested in w_{0b}, we eliminate $w'_{2b} + U_{3a}$ from (48) and (49) and obtain the following differential equation for the determination of w_{0b} (taking $\mathrm{S}_3 \equiv 0$):

$$(50) \qquad \left[w_{0b}^{\prime *\prime} \alpha^3 + 3 \left(w''_{0b} + \sigma \frac{w'_{0b}}{r} \right) \alpha^2 \alpha' \right]^* = \frac{3(1-\sigma^2)\,\mathrm{S}^3}{2E}.$$

For $m = 1$, the equations are $\mathrm{R}^4 = 0$, $\mathrm{Z}_5 = 0$; or

$$(51) \qquad \frac{1}{6(1-\sigma)} w_{0b}^{\prime *\prime *\prime} \alpha^4 + \frac{1}{3(1-\sigma)} \left[\sigma w_{0b}^{\prime *\prime *} (2-\sigma) w_{0b}^{\prime *\prime\prime} \right] \alpha^3 \alpha'$$

$$+ \frac{1}{2(1-\sigma)} \left[(2-\sigma) U_{3a}^{*\prime} - \sigma w_{2b}^{\prime *\prime} \right] \alpha^2 + \frac{\sigma}{1-\sigma} \left[U_{3a}^{*} - w_{2b}^{\prime *} \right] \alpha \alpha'$$

$$+ 2 U'_{3a} \alpha \alpha' - (w'_{4b} + U_{5a}) = \frac{(1+\sigma)(\mathrm{S}^3 - \mathrm{S}_3)\,\alpha'}{E},$$

and, since we take $\mathrm{S}^5 \equiv \mathrm{S}_5 \equiv 0$,

$$(51b) \quad \left\{ \frac{1}{30(1-\sigma)} w_{0b}^{\prime *\prime *\prime} \alpha^5 + \frac{1}{6(1-\sigma)} \left[(2-\sigma) U_{3a}^{*\prime} - \sigma w_{2b}^{\prime *\prime} \right] \alpha^3 \right.$$

$$\left. - (w'_{4b} + U_{5a}) \alpha \right\}^* = 0 = \frac{(1+\sigma)(\mathrm{S}^5 - \mathrm{S}_5)}{E}.$$

From these last two equations we can eliminate $w'_{4b} + U_{5a}$, and obtain an equation involving w_{0b}, w_{2b} and U_{3a}; this equation, together with (48), will determine w_{2b} and U_{3a} individually, w_{0b} having been determined by (50). The foregoing is typical of the general stage in the computation. Let the case $m = 2$ be a final illustration: we obtain two equations from which to eliminate $w'_{6b} + U_{7a}$, and are thus led to a single equation in w_{4b} and U_{5a} which, together with (51), determines these two functions.

(*d*) THE w_{1b} SYSTEM

This system of differential equations is obtained at once from the w_{0b} system by advancing all indices by 1. Since we have $\mathrm{S}_2^2 \equiv \mathrm{S}^4 \equiv \mathrm{S}_4 \equiv \cdots \equiv 0$, the

right-hand members are all 0. In all other respects the equations are those of the w_{0b} system with the indices advanced by 1, and it is quite unnecessary to write them out in full.

Since the U_{1b} and w_{1b} systems are in form the same as the U_{0b} and w_{0b} systems, we see that there are essentially only two distinct types of systems of surface traction differential equations. We shall see later (§ 6) how these two systems are related to the two modes of displacement, radial and axial. With reference to the structure of the differential equations of the two systems, note that when we set $U_{0b} \equiv 0$, $w_{0b} \equiv 0$, and decrease all indices by 2, we get back again to the original U_{0b} and w_{0b} systems. In this connection, we recall the fact that we are dealing with an elastic theory that admits of superposition of solutions.

The striking fact that emerges from the situation just described is that all the differential equations of a system have the same "reduced" linear homogeneous equation. For example, in the w_{0b} system, the function w_{2b} is determined by a differential equation which is found, when computed, to have the same corresponding reduced equation as (50), and similarly for w_{4b}, and so on. This property of the system suggests that once a method is found for integrating the leading equation, all subsequent equations of the system (although the non-homogeneous members become more and more involved) may possibly be integrable by the same method.

It is also interesting to note how closely the reduced equation for the system of axial type† resembles the reduced equation for the system of radial type. Indeed, these reduced equations may be written respectively in the forms

$$(52a) \qquad y^{*'} + 3\left(y' + \sigma \frac{y}{r}\right) \frac{\alpha'}{\alpha} = 0,$$

$$(52b) \qquad y^{*'} + \left(y' + \sigma \frac{y}{r}\right) \frac{\alpha'}{\alpha} = 0.$$

Although we have verified the table for the values $m = 0,1$ only, much light has been shed on the nature of the differential equations, and, incidentally, we have obtained a sufficient number of these equations to enable us to meet the immediate requirements of the applications.

5. **Boundary conditions at an edge.** In (41), we have forms for U and w that satisfy the body force conditions formally, and when the U_a's, U_b's, w_a's, and w_b's satisfy the differential equations of § 4, these forms for U and w also meet the requirements imposed by the surface traction conditions. To complete the program outlined at the beginning of § 3, it remains to determine the constants of integration which enter in the solutions of the differential

† See § 6.

equations. When this has been accomplished by means of the boundary conditions at the edge (or edges) of the plate, U and w are uniquely determined and the particular problem in hand is formally solved. (We shall consistently use the expression "boundary conditions" to refer to boundary conditions at an edge.)

In dealing with boundary conditions, we adopt the conventions of Love, Chapter XXII. For circular plates, an "edge-line" is the intersection of the "middle plane" ($z = 0$) with a *circular* cylinder ($r = r_0$) whose axis is the axis of the plate. The components† of the stress-resultant are denoted by T, N, and the stress-couple is denoted by G. The directions of these components and the conventions with regard to sense‡ are illustrated in Fig. 1. If the normal ν is drawn outwards, we are considering the action of the portion of the plate exterior to $r = r_0$ upon that part of the plate interior to $r = r_0$. A positive T is a radial tension, and points in the direction of the increasing r; similarly, N is directed positively in the sense of the increasing z, and denotes a shearing force normal to the middle plane. The couple G is flexural.

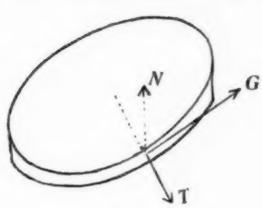

Fig. 1.

The three stress-components are given by the expressions

$$(53a) \qquad T = \int_{a}^{a} t\widehat{rr}\,d\zeta,$$

$$(53b) \qquad N = \int_{-a}^{a} t\widehat{rz}\,d\zeta,$$

$$(53c) \qquad G = \int_{-a}^{a} t^2\widehat{rr}\,\zeta\,d\zeta,$$

where the integrands are expressed in terms of ζ by (16); U and w are given by the series (41), and the U_a's, U_b's, w_a's and w_b's are given by the solutions of the differential equations of §4. If the plate is thick, the traction components have prescribed values at every point of the edge (Love, p. 464); but since

† There is no vector couple in the z direction, and for a circular plate, $S \equiv H \equiv 0$ (cf. Love, p. 56 and p. 141). Thus we are concerned at present only with $T, N\ G$.

‡ Cf. Love, p. 461.

we are considering thin plates (cf. § 16), the tractions applied to the edge may be represented by their force- and couple-resultants estimated per unit of length of the edge-line.

We follow Love (p. 465) in expressing the statical equivalence of the applied tractions, T, N, G, and the stress-resultant and stress-couple components at an edge by equations of the type

$$(54) \qquad T = \mathrm{T}, \quad N = \mathrm{N}, \quad G = \mathrm{G}.$$

Note that the right-hand members are *total* applied tractions. Assuming that integrations with regard to ζ have been performed, we write the applied traction as follows:

$$(55a) \qquad \mathrm{T} = \mathrm{T}_0 + \mathrm{T}_1 t + \mathrm{T}_2 t^2 + \cdots,$$

$$(55b) \qquad \mathrm{N} = \mathrm{N}_0 + \mathrm{N}_1 t + \mathrm{N}_2 t^2 + \cdots,$$

$$(55c) \qquad \mathrm{G} = \mathrm{G}_0 + \mathrm{G}_1 t + \mathrm{G}_2 t^2 + \cdots,$$

where the coefficients are functions of r only, and using (16a) and (16e), we see that relations (54) are identities in t of the form

$$(56a) \qquad \frac{E}{(1+\sigma)(1-2\sigma)} \int_{-a}^{a} \left\{ \sigma\left(U^* t + \frac{\partial w}{\partial \zeta} \right) + (1-2\sigma) U' t \right\} d\zeta \underset{t}{\equiv} \mathrm{T}.$$

$$(56b) \qquad \frac{E}{2(1+\sigma)} \int_{-a}^{a} \left\{ \frac{\partial U}{\partial \zeta} + w' t \right\} d\zeta \underset{t}{\equiv} \mathrm{N},$$

$$(56c) \qquad \frac{E}{(1+\sigma)(1-2\sigma)} \int_{-a}^{a} \left\{ \sigma\left(U^* t + \frac{\partial w}{\partial \zeta} \right) + (1-2\sigma) U' t \right\} t\zeta\, d\zeta \underset{t}{\equiv} \mathrm{G}.$$

Upon equating coefficients of like powers of t, we obtain equations which determine the constants of integration, provided the right-hand members are known at the edges. If the displacements are prescribed at the edges, so that the constants of integration are already known, our equations enable us to compute T, N, G as functions of r of the form (55).

We give below the definitions of certain types of boundary conditions which may obtain at an edge $r = r_0$:

(i) At a "free" edge the requirements are

$$T(r_0) = N(r_0) = G(r_0) = 0. \tag{57}$$

(ii) At a "supported" edge the axial displacement of a point on the middle plane (at the edge-line) is nil, but there is no restriction on the radial displacement of such a point. Also, T and G vanish, so that the complete conditions at a supported edge are

$$w(r_0, 0) = T(r_0) = G(r_0) = 0. \tag{58}$$

(iii) At a "clamped" edge we have

$$U(r_0, 0) = w(r_0, 0) = w'(r_0, 0) = 0; \tag{59}$$

in short, there is neither axial nor radial displacement of a point on the edge-line, and the inclination of the middle plane is not permitted to vary.

To complete our list, we note a type of boundary condition which enters in §§ 7, 8. There both edges of an incomplete plate are subject to radial pressure (or tension), and the boundary conditions take the form

$$T(r_0) = \mathrm{T}(r_0), \qquad T(r_1) = \mathrm{T}(r_1). \tag{60}$$

The applications will require that we have on record a few of the equations obtained from (56) by equating coefficients of like powers of t. From (56a) we find $\mathrm{T}_0 \equiv 0$. From the terms in t we have

$$\frac{2\alpha E}{1-\sigma^2}\left[U'_{0b} + \sigma\frac{U_{0b}}{r}\right] + \frac{2\sigma\alpha}{1-\sigma}\mathrm{S}_0^0 = \mathrm{T}_1. \tag{61a}$$

and we note that T is of order t. Equating the terms in t^2 gives the same result that is obtained by advancing all indices by 1 in (61a); namely

$$\frac{2\alpha E}{1-\sigma^2}\left[U'_{1b} + \sigma\frac{U_{1b}}{r}\right] = \mathrm{T}_2. \tag{61b}$$

In computing the equations from the terms in t^3 and t^4, the integrations with regard to ζ should be carried out, and it is also useful to eliminate w_{3a} and w_{4a} by means of (43 a) and (46). Although these formulas simplify for plates of constant thickness, the general forms are rather forbidding; we have

$$\text{(61 }c\text{)}\quad \frac{2\alpha E}{1-\sigma^2}\Big[U_{2b}'+\sigma\frac{U_{2b}}{r}-\frac{\alpha}{6(1-\sigma)}\{\sigma(4-\sigma)U_{0b}^{*\prime *}\alpha + 6\sigma U_{0b}^{*\prime}\alpha'+(2-\sigma)(1-\sigma)U_{0b}^{*\prime\prime}\alpha\}\Big]=\mathrm{T}_3,$$

$$\text{(61 }d\text{)}\quad \frac{2\alpha E}{1-\sigma^2}\Big[U_{3b}'+\sigma\frac{U_{3b}}{r}-\frac{\alpha}{6(1-\sigma)}\{\sigma(4-\sigma)U_{1b}^{*\prime *}\alpha + 6\sigma U_{1b}^{*\prime}\alpha'+(2-\sigma)(1-\sigma)U_{1b}^{*\prime\prime}\alpha\}\Big]+\frac{\sigma\alpha\mathrm{S}^3}{1-\sigma}=\mathrm{T}_4.$$

Note in (61 c) that U_{0b} is known, so that the essential relationship is between the functions U_{2b} and T_3; a similar remark applies to (61 d). Recall that w_{3a} and w_{4a} are known in terms of U_{2b} and U_{3b}, respectively.

From (56 b), we find $\mathrm{N}_0=\mathrm{N}_1=\mathrm{N}_2=0$. Equating terms in t^3, we have

$$\text{(62 }a\text{)}\qquad w_{0b}^{\prime *\prime}\alpha^3+3\Big[w_{0b}''+\sigma\frac{w_{0b}'}{r}\Big]\alpha^2\alpha' = -\frac{3(1-\sigma^2)\mathrm{N}_3}{2E}.$$

If the plate is of constant thickness ($\alpha=\eta$, $\alpha'=0$), the result is

$$\text{(62 }b\text{)}\qquad w_{0b}^{\prime *\prime} = -\frac{\mathrm{N}_3}{\delta},$$

where

$$\text{(62 }c\text{)}\qquad \delta = \frac{2E\eta^3}{3(1-\sigma^2)};$$

or, eliminating t by means of the relations† $h=\eta t$, $D=\delta t^3$, we may write

$$\text{(62 }d\text{)}\qquad D = \frac{2Eh^3}{3(1-\sigma^2)},$$

† These relations are in agreement with the notation which we first introduced when we wrote $z=\zeta t$ rather than $z=zt$. The Greek letter notation seems to conduce to clearness.

where $2h$ is the actual thickness of the plate and D is the "flexural rigidity" (Love, p. 470).

The equation in N_4 is obtained at once from (62a) by advancing the indices by 1. Equating terms in t^5, we find (retaining only even powers of ζ in the integrand)

$$(62e)\quad \frac{E}{2(1+\sigma)}\int_{-\alpha}^{\alpha}\left\{-\frac{1}{6(1-\sigma)}w_{0b}^{\prime *\prime *\prime}\zeta^4+\frac{\sigma}{2(1-\sigma)}w_{2b}^{\prime *\prime}\zeta^2 -\frac{2-\sigma}{2(1-\sigma)}U_{3a}^{*\prime}\zeta^2+w_{4b}'+U_{5a}\right\}d\zeta = N_5.$$

From this equation we can eliminate $w_{4b}'+U_{5a}$ by means of (51) and U_{3a} by means of (48); and integrating, we obtain a relation between w_{2b} and N_5, since w_{0b} is already known. Since the relation is rather involved, we shall be content to write it out only for the special case $\alpha'=0$; (62e) then reduces to

$$(62f)\quad w_{2b}^{\prime *\prime}-\frac{3\sigma-8}{10(1-\sigma)}w_{0b}^{\prime *\prime *\prime}\eta^2=-\frac{N_5}{\delta}.$$

We turn finally to (56c) and find $G_0\equiv G_1\equiv G_2\equiv 0$; it appears that G, as well as N, must be of order t^3. We find

$$(63a)\quad \left[w_{0b}''+\sigma\frac{w_{0b}'}{r}\right]\alpha^3=-\frac{3(1-\sigma^2)}{2E}G_3,$$

and a similar equation in G_4 obtained by advancing indices by 1.

The terms in t^5 give us

$$(63b)\quad \frac{E}{1+\sigma}\int_{-\alpha}^{\alpha}\left\{\frac{\sigma}{6(1-\sigma)}\left[w_{0b}^{\prime *\prime *}\zeta^3-3w_{2b}^{\prime *}\zeta+3U_{3a}^{*}\zeta\right] +\left[\frac{2-\sigma}{6(1-\sigma)}w_{0b}^{\prime *\prime\prime}\zeta^3+U_{3a}'\zeta\right]\right\}\zeta\,d\zeta=G_5,$$

and when we eliminate U_{3a} by means of (48) and integrate, we have the desired relation between w_{2b} and G_5. If $\alpha'=0$, we find

$$(63c)\quad w_{2b}''+\sigma\frac{w_{2b}'}{r}+\frac{\eta^2}{10(1-\sigma)}\left[\sigma(4+\sigma)w_{0b}^{\prime *\prime *}+(1-\sigma)(8+\sigma)w_{0b}^{\prime *\prime\prime}\right]=-\frac{G_5}{\delta}.$$

We have obtained enough of the relations of type (56) to enable us to deal with the simpler applications. If T, N, G are initially prescribed, the series (55) are either nil or else terminate with the leading term; for example: $T \equiv T_1 t$, $N \equiv N_3 t^3$, $G \equiv G_3 t^3$. When these components are unknown elements and the plate is of constant thickness, the series (55), as well as (6), will terminate provided the distribution of load meets the requirements of § 13. But for a random type of loading, or in the case of an initially prescribed central deflection, and obviously in cases of variable thickness, T, N, G, as well as U, w, are given in general by infinite series.

6. **Analysis of the formulas of displacement by the aid of certain transformations.** The formulas (41) for U and w, together with the differential equations of § 4, are quite general; they enable us to handle, for example, a problem involving simultaneous radial and axial pressure. Before dealing with the applications, it is important to ascertain what simplifications occur in (41) when we consider

(i) the case of radial pressure only.

(ii) the case of axial pressure only.

In this connection, we make use of the following transformations:

$$(t, -t), (w, -w), \tag{64a}$$

$$(\zeta, -\zeta), (w, -w), (\boldsymbol{\alpha}, -\boldsymbol{\alpha}), \tag{64b}$$

$$(t, -t), (U, -U), \tag{64c}$$

$$(\zeta, -\zeta), (U, -U), (\boldsymbol{\alpha}, -\boldsymbol{\alpha}), \tag{64d}$$

where $(t, -t)$ signifies the replacement of t by $-t$, etc.

Clearly each of these transformations turns the plate upside down and at the same time reverses the signature of either w or U. When one of these transformations does not disturb the conditions that characterize a given problem, it cannot influence the solution, and hence may be used to simplify the formulas of displacement.

If the transformations (64) are to leave U and w unchanged, they must leave unaltered the fundamental relations (14), (21), (56) which determine U and w.† Since $F_r \equiv F_z \equiv 0$, all requirements are met in the case of (14).

† We need not concern ourselves here with the fact that the displacements themselves may be prescribed at the edges, as in (59). In such cases we discern by direct examination of (64) the transformations which do not disturb the boundary conditions.

If (21) and (56) are to remain unchanged under (64), we find the following restrictions on S, T, G, N, where S refers to both S^{ν} and S_{ν}:†

(65a) $S(t) = S(-t),\quad T(t) = -T(-t),$

$N(t) = N(-t),\quad G(t) = G(-t);$

(65b) $N \equiv G \equiv 0;$

(65c) $S(t) = -S(-t),\quad T(t) = T(-t),$

$N(t) = -N(-t),\quad G(t) = -G(-t);$

(65d) $S \equiv T \equiv 0.$

Here the order of arrangement is the same as that in (64), and the conditions are concerned with the vanishing of certain terms in the expansions (18) and (55).

(a) The radial case

Under this heading we consider the pair of transformations (64a) and (64b), assuming that (65a) and (65b) do not conflict with the boundary conditions, so that U and w are unaltered. Using (6), we find that these transformations yield respectively the following relations:

(66a) $U_{2m+1}(r, \zeta) \equiv w_{2m}(r, \zeta) \equiv 0 \qquad (m = 0, 1, 2, \ldots).$

(66b) $U_m(r, \zeta) - U_m(r, -\zeta) \equiv w_m(r, \zeta) + w_m(r, -\zeta) \equiv 0$

$(m = 0, 1, 2, \ldots).$

Turning to (41), we observe that (66a) implies the vanishing of all the unknowns of the w_{0b} and U_{1b} systems, and that (66b) demands the vanishing of all the unknowns of the w_{1b} system; for (66) demands that each of these unknowns equal its negative, and since they are given uniquely by the differential equations of § 4 (provided (65a) and (65b) do not conflict with the boundary conditions), this cannot happen except they vanish identically. With only the unknowns of the U_{0b} system remaining, the formulas (41) simplify tremendously. For future reference, the results are recorded below:

† In deriving (65b) and (65d), note that the right-hand members of (56) contain a multiplicative factor $2a$ which changes sign under (64b) and (64d).

$$(67a)\quad U = U_{0b} + \left[U_{2b} - \frac{2-\sigma}{2(1-\sigma)} U_{0b}^{*\prime} \zeta^2\right] t^2$$

$$+ \left[U_{4b} - \frac{1}{2(1-2\sigma)} w'_{3a} \zeta^2 - \frac{1-\sigma}{1-2\sigma} U_{2b}^{*\prime} \zeta^2 + \frac{3-\sigma}{24(1-\sigma)} U_{0b}^{*\prime *\prime} \zeta^4\right] t^4$$

$$+ \left[U_{6b} - \frac{1}{2(1-2\sigma)} w'_{5a} \zeta^2 - \frac{1-\sigma}{1-2\sigma} U_{4b}^{*\prime} \zeta^2 + \frac{1}{12(1-2\sigma)} w_{3a}^{\prime *\prime} \zeta^4\right.$$

$$\left. + \frac{3-2\sigma}{24(1-2\sigma)} U_{2b}^{*\prime *\prime} \zeta^4 - \frac{4-\sigma}{720(1-\sigma)} U_{0b}^{*\prime *\prime *\prime} \zeta^6\right] t^6 + \cdots,$$

$$(67b)\quad w = w_{1a} \zeta t + \left[w_{3a} \zeta + \frac{1+\sigma}{6(1-\sigma)} U_{0b}^{*\prime *} \zeta^3\right] t^3$$

$$+ \left[w_{5a} \zeta + \frac{\sigma}{3(1-2\sigma)} w_{3a}^{\prime *} \zeta^3 + \frac{1}{6(1-2\sigma)} U_{2b}^{*\prime *} \zeta^3\right.$$

$$\left. - \frac{2+\sigma}{120(1-\sigma)} U_{0b}^{*\prime *\prime *} \zeta^5\right] t^5 + \cdots.$$

Note that $w(r, 0) = 0$; that is, the middle plane is stretched without deflection at any point. It is interesting to interpret physically the transformations (64a) and (64b); we observe that when $(t, -t)$ or $(\zeta, -\zeta)$, $(\alpha, -\alpha)$ turns the plate upside down, the requirement $(w, -w)$ takes care of the necessary reversal of sign in the axial displacement. The advantage of deducing formulas (67) at this point will appear when we arrive at the applications of §§ 7, 8.

(b) The axial case

From (6), we find that the pair of transformations (64c) and (64d) yield respectively the relations

$$(68a)\quad U_{2m}(r, \zeta) \equiv w_{2m+1}(r, \zeta) \equiv 0 \qquad (m = 0, 1, 2, \ldots),$$

$$(68b)\quad U_m(r, \zeta) + U_m(r, -\zeta) \equiv w_m(r, \zeta) - w_m(r, -\zeta) \equiv 0 \qquad (m = 0, 1, 2, \ldots).$$

By (41), the first of these relations implies the vanishing of all the unknowns of the U_{0b} and w_{1b} systems, and (68b) shows that all the unknowns of the U_{1b} system must vanish also. The argument here is the same as in the radial case. We are assuming, of course, that (65c) and (65d) do not conflict with the boundary conditions.

Before writing down any formulas of displacement for the axial case, let us see how much we can infer from (64d) when $S^r \equiv S^3 t^3 \not\equiv 0$, $S_r \equiv 0$.

Although the conditions of $(65d)$ are no longer met, an examination of the differential equations of the U_{1b} system shows that the only unknowns of the system whose vanishing we cannot now infer are U_{3b} and w_{4a}. Since the special case we are considering is one of vital importance in the applications, we shall retain these two unknowns in our formulas of displacement for the axial case.

The physical situation here is of considerable interest. When U and w are respectively even and odd functions of ζ, as in the radial case, or respectively odd and even functions of ζ, as in the pure† axial case, there are obvious symmetrical relations governing the form into which an initially vertical filament is bent. But if the upper base of the plate is loaded, the terms in U_{3b} and w_{4a} enter, and U and w are no longer respectively odd and even functions of ζ; not only are certain changes reflected in the form into which the initially vertical filament is bent, but also there is the new possibility that the middle plane is *extended* as well as bent.

The inclusion of the unknowns U_{3b} and w_{4a} gives us the following formulas of displacement in the axial case:

$$(69a)\qquad U = -w'_{0b}\zeta t + \left[U_{3b} + U_{3a}\zeta + \frac{2-\sigma}{6(1-\sigma)} w'^{*\prime}_{0b}\zeta^3\right] t^3$$

$$+ \left[U_{5a}\zeta - \frac{1}{2(1-2\sigma)} w'_{4a}\zeta^2 - \frac{1-\sigma}{1-2\sigma} U^{*\prime}_{3b}\zeta^2 - \frac{3-2\sigma}{12(1-\sigma)} U^{*\prime}_{3a}\zeta^3\right.$$

$$\left. + \frac{1}{12(1-\sigma)} w'^{*\prime}_{2b}\zeta^3 - \frac{3-\sigma}{120(1-\sigma)} w'^{*\prime *\prime}_{0b}\zeta^5\right] t^5 + \cdots,$$

$$(69b)\qquad w = w_{0b} + \left[w_{2b} + \frac{\sigma}{2(1-\sigma)} w'^{*}_{0b}\zeta^2\right] t^2 + \left[w_{4b} + w_{4a}\zeta\right.$$

$$\left. - \frac{1}{4(1-\sigma)} - U^{*}_{3a}\zeta^2 - \frac{1-2\sigma}{4(1-\sigma)} w'^{*}_{2b}\zeta^2 - \frac{1+\sigma}{24(1-\sigma)} w'^{*\prime *}_{0b}\zeta^4\right] t^4$$

$$+ \left[w_{6b} - \frac{1}{4(1-\sigma)} U^{*}_{5a}\zeta^2 - \frac{1-2\sigma}{4(1-\sigma)} w'^{*}_{4b}\zeta^2 + \frac{\sigma}{3(1-2\sigma)} w'^{*}_{4a}\zeta^3\right.$$

$$+ \frac{1}{6(1-2\sigma)} U^{*\prime *}_{3b}\zeta^3 + \frac{1}{24(1-\sigma)} U^{*\prime *}_{3a}\zeta^4 - \frac{\sigma}{24(1-\sigma)} w'^{*\prime *}_{2b}\zeta^4$$

$$\left. + \frac{2+\sigma}{720(1-\sigma)} w'^{*\prime *\prime *}_{0b}\zeta^6\right] t^6 + \cdots.$$

† We agree that (65*c*) sufficiently characterizes the axial case. If both (65*c*) and (65*d*) obtain, we have the *pure* axial case. An analogous distinction might be made in the radial case; then (67) would refer to the *pure* radial case.

If there are no surface tractions ($S^3 = 0$), we suppress the terms involving U_{3b} and w_{4a} and find $U(r, 0) = 0$; that is, the middle plane is bent without extension. Examples of this type are treated in §§ 9, 10. In the examples of §§ 11—15 we have $S^3 \neq 0$, and the terms in U_{3b} and w_{4a} must be retained; indeed it is precisely these terms which provide in the second of these examples (§ 12) for a uniform stretching of the middle plane.

Considering only the leading terms in (69), we see that *to a first approximation* a linear element originally normal to $z = 0$ remains straight in the strained state, is not compressed transversely, and also meets at right angles the surface into which the middle plane is bent.

In the axial case it is particularly interesting to form the physical picture for transformations (64 *a*) and (64 *d*), and observe how, after the plate has been turned upside down, the displacements are still correctly given by virtue of $(U, -U)$.

We observe that the U_{0b} system predominates in the radial case, with the w_{1b} system entering as an auxiliary system of axial type. The situation is reversed in the axial case; here the w_{0b} system comes first, and the U_{1b} system of radial type provides for radial displacements of higher order than the axial displacements. In the *pure* radial and axial cases the auxiliary w_{1b} and U_{1b} systems disappear. Of course the solutions of the radial and axial cases may be superposed if desired; that is, the most general problem may be solved by a synthesis of solutions of radial and axial type.

It may be noted that the energy integral,

$$(69c)\quad \frac{\pi E}{2(1+\sigma)(1-2\sigma)} \int_{r_1}^{r_0} \int_{-a}^{a} \left\{ 2(1-\sigma) \left[U^* + \frac{1}{t} \frac{\partial w}{\partial \zeta} \right]^2 t \right.$$
$$\left. + (1-2\sigma) \left[\left(\frac{1}{t} \frac{\partial U}{\partial \zeta} + w' \right)^2 t - 4 \frac{UU'}{r} t - 4U \frac{\partial w}{\partial \zeta} \right] \right\} r \, dr \, d\zeta,$$

is of order t in the radial case and of order t^3 in the axial case, in agreement with the known facts.

Our formulas are in general power series in both z and h. When $z = 0$, we have power series in h alone. Also the applied tractions, (18) and (55), are given by power series in h alone.

Although we take the point of view of *tractions* applied at the faces of the plate, there is the possibility of prescribing the *displacements* at the faces and subsequently computing the corresponding applied tractions. So long as the prescribed elements define a determinate problem, it seems clear that the method of series is applicable.

Finally, we note that t is not necessarily a *small* parameter. What is important is that the actual plate under consideration be thin; that is, when the parameter t is suppressed, the ratio of the thickness of the plate to its diameter must be in general† small.

Further remarks will be found in a concluding paragraph which follows the applications.

PART II. APPLICATIONS

7. Incomplete circular plate under internal and external radial pressure.‡ As an example of the radial case, with the displacements given by (67), consider a plate of constant thickness with outer and inner radii r_0 and r_1 respectively. In the notation agreed upon, the thickness is $2a = 2h$ or $2\alpha t = 2\eta t$.

If the pressures (per unit of edge-line arc) are p_0 and p_1 on the external and internal boundaries respectively, we have

$$\text{(70}a\text{)} \qquad \mathrm{T}(r_0) = \mathrm{T}_1(r_0)\,t = -\int_{-\eta}^{\eta} p_0\,t\,d\zeta = -2\,\eta\,t\,p_0.$$

$$\text{(70}b\text{)} \qquad \mathrm{T}(r_1) = \mathrm{T}_1(r_1)\,t = -\int_{-\eta}^{\eta} p_1\,t\,d\zeta = -2\,\eta\,t\,p_1,$$

and assuming $\mathrm{S}_0^0 = 0$, we have, by (61 a),

$$\text{(71}\,a\text{)} \qquad \frac{E}{1-\sigma^2}\left[U'_{0b} + \sigma\frac{U_{0b}}{r}\right]_{r=r_0} = -p_0,$$

$$\text{(71}\,b\text{)} \qquad \frac{E}{1-\sigma^2}\left[U'_{0b} + \sigma\frac{U_{0b}}{r}\right]_{r=r_1} = -p_1.$$

The first differential equation to be solved is (42). When $\alpha' = 0$, this equation reduces to

$$\text{(72}a\text{)} \qquad U^{*\prime}_{0b} = 0.$$

† When the method leads to results which hold for a thick plate (see Part II), we have an exceptional case.

‡ The solution of essentially the same problem is found in Love (p. 141); in point of fact, the applications of §§ 9—12 to plates of constant thickness do not lead to results that are new. It is enough that these examples serve admirably to illustrate the directness of the present method, as well as to suggest the detailed manner of procedure in cases that are new, notably those of variable thickness.

and its complete solution is

$$U_{0b} = C_1 r + C_2 r^{-1}. \tag{72b}$$

The constants are determined by (71), and have the values

$$C_1 = \frac{1-\sigma}{E}\left[\frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2}\right], \tag{73a}$$

$$C_2 = \frac{1+\sigma}{E}\left[\frac{(p_1 - p_0) r_0^2 r_1^2}{r_0^2 - r_1^2}\right]. \tag{73b}$$

All remaining unknowns appearing in (67) may be shown to vanish identically. For example, equations (43a) and (44) reduce to

$$\sigma U_{2b}^{*} + (1-\sigma) w_{3a} = 0, \tag{74a}$$

$$(1-\sigma) U_{2b}^{*\prime} + w_{3a}' = 0; \tag{74b}$$

hence $U_{2b}^{*\prime} = 0$, or

$$U_{2b} = C_1' r + C_2' r^{-1}. \tag{74c}$$

But (61c) gives

$$\left[U_{2b}' + \sigma \frac{U_{2b}}{r}\right]_{r=r_0} = 0. \tag{74d}$$

$$\left[U_{2b}' + \sigma \frac{U_{2b}}{r}\right]_{r=r_1} = 0, \tag{74e}$$

and hence $U_{2b} \equiv 0$. Also, $w_{3a} \equiv 0$.

Since all unknowns vanish save U_{0b}, the complete solution, as given by (67), is the following:

$$U = \frac{1-\sigma}{E}\left[\frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2}\right] r + \frac{1+\sigma}{E}\left[\frac{(p_1 - p_0) r_0^2 r_1^2}{r_0^2 - r_1^2}\right] \frac{1}{r}, \tag{75a}$$

$$w = -\frac{2\sigma}{E}\left[\frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2}\right] z. \tag{75b}$$

where we have made the return to the original variable z. From (75b), we see that there is uniform axial extension of amount given by the coefficient of z.

If the thickness of the plate is to remain constant, it is necessary to impose a tension. Since there is no axial extension, w_{1a} must vanish; or, by (33b),

(76)
$$U_{0b}^{*} = \frac{(1+\sigma)(1-2\sigma)\,S_0^0}{\sigma E}.$$

This relation may be used to eliminate S_0^0 from (61a), so that our boundary conditions become

(77a)
$$\frac{E}{(1+\sigma)(1-2\sigma)}\left[(1-\sigma)\,U_{0b}' + \sigma\frac{U_{0b}}{r}\right]_{r=r_0} = -p_0.$$

(77b)
$$\frac{E}{(1+\sigma)(1-2\sigma)}\left[(1-\sigma)\,U_{0b}' + \sigma\frac{U_{0b}}{r}\right]_{r=r_1} = -p_1.$$

Taking U_{0b} in the form

(78)
$$U_{0b} = K_1 r + K_2 r^{-1},$$

we obtain from (77) the following values for the constants:

(79a)
$$K_1 = \frac{(1+\sigma)(1-2\sigma)}{E}\left[\frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2}\right],$$

(79b)
$$K_2 = C_2;$$

and we have for the complete solution in the case of no axial extension

(80a)
$$U = \frac{(1+\sigma)(1-2\sigma)}{E}\left[\frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2}\right] r + \frac{1+\sigma}{E}\left[\frac{(p_1-p_0)\,r_0^2 r_1^2}{r_0^2 - r_1^2}\right]\frac{1}{r},$$

(80b) $w = 0$.

Having determined the constants in (78), we may compute U_{0b}^{*} and obtain from (76) the value of the imposed tension. We find

(81a)
$$S_0^0 = \frac{\sigma E}{(1+\sigma)(1-2\sigma)}\,U_{0b}^{*} = \frac{2\sigma(p_1 r_1 - p_0 r_0^2)}{r_0^2 - r_1^2}.$$

In the first case considered ($S_0^0 = 0$), let us suppose that $p_0 = 0$; then we have $\widehat{zz} = \widehat{zr} = 0$, and

$$\widehat{rr} = \frac{p_1 r_1^2}{r_0^2 - r_1^2}\left[1 - \frac{r_0^2}{r^2}\right], \tag{81b}$$

$$\widehat{\theta\theta} = \frac{p_1 r_1^2}{r_0^2 - r_1^2}\left[1 + \frac{r_0^2}{r^2}\right]. \tag{81c}$$

We observe that the circumferential tension is the greatest tension; its value at $r = r_1$ exceeds by p_1 its value at $r = r_0$. The greatest extension is the circumferential extension ($e_{\theta\theta}$) at $r = r_1$.

If $r_0 - r_1$ is small in comparison with the thickness $2h$, we have results that obtain for a tube rather than for a plate (cf. Love, loc. cit.).

Finally, attention may be called to the fact that we might have prescribed the radial displacements of the inner and outer edges and computed the corresponding pressures.

8. **Incomplete circular plate of variable thickness under internal and external radial pressure**. The displacements are given by (67), and the first equation to integrate is (42). Assuming the bases free from traction ($S_0^0 = 0$), equation (42) becomes

$$U_{0b}^{*\prime}\,\alpha + \left[U'_{0b} + \sigma\,\frac{U_{0b}}{r}\right]\alpha' = 0. \tag{82}$$

If we restrict α to be of the form

$$\alpha(r) = \eta\, r^k\; [a(r) = h r^k,\; h = \eta t]. \tag{83}$$

equation (82) is of Euler's type and is readily transformed by a change of independent variable into a linear equation with constant coefficients.† When $k = 0$, we have the plate of uniform thickness already considered; but in general, the thickness at the center is either infinite or nil. We avoid this difficulty by considering an incomplete plate.

When $k = -1$, $h = 1$, the plate is as shown below in the figure. Note that the choice of h determines the thinness of the plate. We do not specialize k in the discussion that follows.

† Cf. Goursat-Hedrick, *Mathematical Analysis*, vol. II, part II, p. 123.

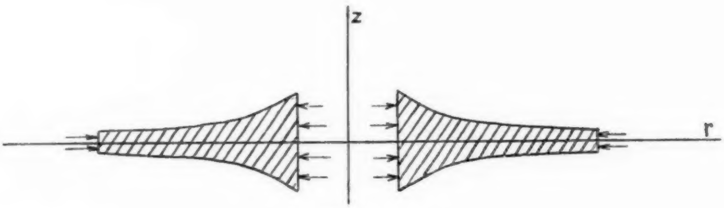

Fig. 2.

Since the ratio α'/α is independent of η, the leading terms in the displacements are independent of the thickness of the plate, and this is explainable when we recall the manner in which the pressures at the edges vary with the thickness.† But equations (43a) and (44) are altered by a change in η, and we see that subsequent terms in the displacements will not be independent of the thickness of the plate.

Using (83), equation (82) becomes

$$(84) \qquad U''_{0b} + (1+k)\frac{U'_{0b}}{r} - (1-\sigma k)\frac{U_{0b}}{r^2} = 0,$$

and the solution of this equation is

$$(85a) \qquad U_{0b} = C_1 r^{m_1} + C_2 r^{m_2},$$

where m_1 and m_2 are roots of the characteristic equation

$$(85b) \qquad m^2 + km - (1-\sigma k) = 0.$$

The discriminant of (85b) is $\Delta = k^2 + 4(1-\sigma k)$, and vanishes when $k = 2\sigma \pm 2\sqrt{\sigma^2-1}$; but since $\sigma < 1$ (Love, p. 103), this value of k is never real. Thus we see that in any *bona fide* physical problem, the roots of the characteristic equation are real and distinct.

If the respective pressures on the outer and inner edges are p_0, p_1, the conditions at the boundary are given by (71). These equations determine the constants in (85a), and we readily find

† Cf. §§ 5, 16.

$$(86)\qquad U_{0b} = \frac{1-\sigma^2}{E(r_0^{m_1} r_1^{m_2} - r_0^{m_2} r_1^{m_1})}\left[\frac{r^{m_1}}{m_1+\sigma}(p_1 r_0^{m_2} r_1 - p_0 r_1^{m_2} r_0) - \frac{r^{m_2}}{m_2+\sigma}(p_1 r_0^{m_1} r_1 - p_0 r_1^{m_1} r_0)\right].$$

If $k = 0$, then $m_1 = 1$, $m_2 = -1$. and (86) reduces to the U_{0b} of the preceding paragraph.

By virtue of (86) and (33*b*), we now know the leading terms in the displacements; that is,

$$(87a)\qquad U = U_{0b} + \cdots.$$

$$(87b)\qquad w = -\frac{\sigma}{1-\sigma} U_{0b}^{*} z + \cdots.$$

If we wish to compute further terms in U and w, we consider next the equations (43*a*) and (44). Eliminating w_{3a}, we obtain the following equation for the determination of U_{2b}:

$$(88a)\qquad U_{2b}^{*\prime}\, \alpha + \left[U_{2b}' + \sigma\,\frac{U_{2b}}{r}\right]\alpha' = \Phi\,(U_{0b},\ \alpha,\ \text{and their derivatives}).$$

Since U_{0b} and α are known, Φ is a known† function of r, and this equation has a solution of the form

$$(88b)\qquad U_{2b} = C_1' r^{m_1} + C_2' r^{m_2} + \varphi\,(r),$$

where φ is the particular integral. By (43*a*), the function w_{3a} is also known up to C_1' and C_2'. By means of (61*c*), where $T_3(r_0) = T_3(r_1) = 0$, the constants of integration are determined, and since there is no longer the homogeneity‡ that characterized the case $k = 0$, these constants are in general different from 0. The situation here is typical for what happens as we continue to determine subsequent terms in U and w, and it is clear that if the thickness is variable, the series for the displacements will not in general terminate.

† From (43), (44), (61*c*), we ascertain in particular the manner in which α and its derivatives enter in Φ, and verify the fact that U_{2b} contains the factor η^2. Thus the term of order t^2 in (67*a*) is seen to be homogeneous of order 2 in ζ and η, and in general the coefficient of t^n is homogeneous of order n in ζ and η, as pointed out in § 1.

‡ Cf. (74*d*) and (74*e*).

9. **Complete circular plate loaded at its center and clamped at the edge.** In this paragraph and the next three, we continue to deal with plates of constant thickness. Although we do not obtain new results, we show how the present method lends itself to direct computation of the solutions in question. Indeed, it seems desirable that in exhibiting the power of the method as applied to plates of constant thickness, we should begin with familiar rather than with new examples.

Under central load, the surface tractions are nil, and the displacements are given by (69) without the terms in U_{3b} and w_{4a}. The boundary conditions are given by (59).

The first differential equation to be considered is (50), and since $\alpha' = 0$, it reduces to

$$w_{0b}^{\prime *\prime *} = 0, \tag{89a}$$

or

$$w_{0b}^{\prime *\prime} = \frac{K}{r}. \tag{89b}$$

The constant K is determined by (62b). If W is the central load, we have

$$W = 2\pi r N_3 t^3, \tag{90}$$

since $2\pi r N$ is the resultant shearing force on the portion of the plate lying within a circle of radius r. Thus, (89b) becomes

$$w_{0b}^{\prime *\prime} = -\frac{W}{2\pi r \delta t^3}, \tag{91}$$

and the shear is infinite at the center. Integrating, we find

$$w_{0b} = \frac{W}{8\pi D}\left[r^2 \log\frac{r_0}{r} + r^2\right] + C_1 r^2 + C_2 + C_3 \log r, \tag{92}$$

where D is the flexural rigidity. Since the plate is complete up to the center, C_3 must vanish. The constants C_1 and C_2 are determined by the conditions $w_{0b}(r_0) = w_{0b}'(r_0) = 0$, which follow from (59). We find

$$w_{0b} = -\frac{W}{8\pi D}\left[\frac{1}{2}(r_0^2 - r^2) - r^2 \log\frac{r_0}{r}\right]. \tag{93}$$

We turn next to the determination of w_{2b} and U_{3a}. From (48), we have

$$w'_{2b} + U_{3a} = \frac{W\eta^2}{2\pi r D(1-\sigma)}. \tag{94}$$

and equations (51) and (52) reduce to

$$\frac{1}{2(1-\sigma)}[(2-\sigma)U^{*\prime}_{3a} - \sigma w'^{*\prime}_{2b}]\eta^2 - (w'_{4b} + U_{5a}) = 0. \tag{95a}$$

$$-\frac{\sigma}{6(1-\sigma)} w'^{*\prime *}_{2b}\, \eta^2 + \frac{2-\sigma}{6(1-\sigma)} U^{*\prime *}_{3a}\, \eta^2 - (w'_{4b} + U_{5a})^* = 0. \tag{95b}$$

Eliminating $w'_{4b} + U_{5a}$ from the last two equations, and using (94) to get rid of U_{3a}, the result is

$$w'^{*\prime *}_{2b} = 0, \tag{96a}$$

or

$$w^{*\prime *}_{2b} = \frac{K'}{r}. \tag{96b}$$

But $N_5 \equiv 0$ in (62*f*), and $K' = 0$; hence we find, upon integration and suppression of the term in log r.

$$w_{2b} = C'_1 r^2 + C'_2. \tag{96c}$$

Finally, the boundary conditions $w_{2b}(r_0) = w'_{2b}(r_0) = 0$ give us $w_{2b} \equiv 0$. and returning to (94) we have

$$U_{3a} = \frac{W\eta^2}{2\pi r D(1-\sigma)}. \tag{97}$$

The conditions $N_7 \equiv 0$ and $w_{4b}(r_0) = w'_{4b}(r_0) = 0$ give $w_{4b} \equiv 0$. From (95*a*) the equation analogous to (94) is

$$w'_{4b} + U_{5a} = 0, \tag{98}$$

so that the analogue to (97) is $U_{5a} \equiv 0$. A similar argument shows that all remaining unknowns vanish identically.

We are now in a position to write down the formulas of displacement (69). Making the return to the original variable z, we have

$$(99a)\qquad U = -\frac{Wrz}{4\pi D}\log\frac{r_0}{r} + \frac{W}{12\pi D(1-\sigma)r}\left[6h^2z-(2-\sigma)z^3\right],$$

$$(99b)\qquad w = -\frac{W}{8\pi D}\left[\tfrac{1}{2}(r_0^2-r^2)-r^2\log\frac{r_0}{r}\right] + \frac{\sigma W z^2}{8\pi D(1-\sigma)}\left[2\log\frac{r_0}{r}-1\right].$$

The central deflection is

$$(100)\qquad d = w(0,0) = -\frac{Wr_0^2}{16\pi D},$$

and by means of this relation it is possible to express the displacements in terms of an initially prescribed central deflection. Indeed, it is possible to solve the problem from the point of view of an initially prescribed central deflection and subsequently compute the load in terms of d. We follow this suggestion in the next paragraph and obtain some interesting results.

We observe that the terms of higher order in (99) become infinite at the center.† But for a thin plate, the formulas yield reasonable displacements outside a certain small neighborhood of the axis. If $z=0$, we have the formula of the approximate theory (Love, p. 494). A meridian cut of the surface into which the middle plane is bent has infinite curvature at $r=0$.

10. Complete circular plate loaded at its center and supported at the edge. As in the preceding paragraph, we have

$$(101)\qquad w_{0b} = \frac{W}{8\pi D}\left[r^2\log\frac{r_0}{r}+r^2\right]+C_1r^2+C_2;$$

but the constants are to be determined by boundary conditions following from (58), namely

$$(102)\qquad w_{0b}(r_0) = G_3(r_0) = 0.$$

Applying these conditions, we obtain

$$(103)\qquad w_{0b} = -\frac{W}{16\pi D}\left[\frac{3+\sigma}{1+\sigma}(r_0^2-r^2)-2r^2\log\frac{r_0}{r}\right].$$

† See the remark at the close of § 10.

Turning to the unknowns w_{2b} and U_{3a}, we find again an equation of the form (96c), and since $w_{2b}(r_0) = 0$, we have

$$w_{2b} = K(r^2 - r_0^2). \tag{104}$$

To determine K, we use (63c), where $G_5(r_0) = 0$. When w_{2b} is known, U_{3a} is determined by (48), and the final results are as follows:

$$w_{2b} = -\frac{(8+\sigma)W\eta^2}{40(1+\sigma)\pi D r_0^2}(r^2 - r_0^2), \tag{105a}$$

$$U_{3a} = \frac{(8+\sigma)W\eta^2 r}{20(1+\sigma)\pi D r_0^2} + \frac{W\eta^2}{2(1-\sigma)\pi D r}. \tag{105b}$$

From $w_{4b}(r_0) = G_7(r_0) = 0$, we find $w_{4b} \equiv U_{5a} \equiv 0$, and since all further unknowns vanish identically, we may write down the following complete solution:

$$\begin{aligned} U = {} & -\frac{Wr}{8\pi D}\left[2\log\frac{r_0}{r} - 1 + \frac{3+\sigma}{1+\sigma}\right]z \\ & + \frac{W}{4\pi D}\left[\frac{(8+\sigma)rh^2 z}{5(1+\sigma)r_0^2} + \frac{2h^2 z}{(1-\sigma)r} - \frac{(2-\sigma)z^3}{3(1-\sigma)r}\right], \end{aligned} \tag{106a}$$

$$\begin{aligned} w = {} & -\frac{W}{16\pi D}\left[\frac{3+\sigma}{1+\sigma}(r_0^2 - r^2) - 2r^2\log\frac{r_0}{r}\right] \\ & + \frac{W}{8\pi D}\left[\frac{(8+\sigma)(r_0^2 - r^2)h^2}{5(1+\sigma)r_0^2} + \frac{\sigma}{1-\sigma}\left\{2\left(\log\frac{r_0}{r} - 1\right) + \frac{3+\sigma}{1+\sigma}\right\}z^2\right] \\ & - \frac{\sigma(8+\sigma)Wh^2 z^2}{20(1-\sigma^2)\pi D r_0^2}. \end{aligned} \tag{106b}$$

The central deflection is

$$d = -\frac{(3+\sigma)Wr_0^2}{16(1+\sigma)\pi D}\left[1 - \frac{2(8+\sigma)h^2}{5(3+\sigma)r_0^2}\right]. \tag{107}$$

Solving for W in terms of d, we find

$$(108a)\qquad W = -\frac{16(1+\sigma)\pi Dd}{(3+\sigma)r_0^2}\left[\frac{1}{1-\dfrac{2(8+\sigma)h^2}{5(3+\sigma)r_0^2}}\right].$$

Substituting this value in (106), we obtain formulas of displacement in terms of d, and these are closed forms; that is, when the parameter t is introduced, these formulas of displacement are not power series in t. But by expanding (108a) in series,

$$(108b)\qquad W = -\frac{16(1+\sigma)\pi Dd}{(3+\sigma)r_0^2}\left[1+\frac{2(8+\sigma)h^2}{5(3+\sigma)r_0^2}+\cdots\right],$$

we obtain for U and w the same formal developments in t which result when the problem is solved from the point of view of an initially prescribed central deflection.† If we were not to write out in full these series for U and w in terms of the central deflection, we might fail to observe a situation that obtains in all these cases, namely, the coefficient of t^n, considered as a function of r and r_0, is homogeneous‡ of order $-n$. We have

$$(109a)\qquad U = \frac{2dr}{r_0^2}\left[\frac{1+\sigma}{3+\sigma}\left(2\log\frac{r_0}{r}-1\right)+1\right]z$$
$$+\frac{4(1+\sigma)dr}{5(3+\sigma)r_0^4}\left[\left\{\frac{8+\sigma}{3+\sigma}\left(2\log\frac{r_0}{r}-1\right)-\frac{10r_0^2}{(1-\sigma)r^2}\right\}h^2z\right.$$
$$\left.+\frac{5(2-\sigma)r_0^2}{3(1-\sigma)r^2}z^3\right]+\cdots,$$

$$(109b)\qquad w = \frac{d}{r_0^2}\left[(r_0^2-r^2)-\frac{2(1+\sigma)}{3+\sigma}r^2\log\frac{r_0}{r}\right]$$
$$-\frac{4(1+\sigma)d}{5(3+\sigma)r_0^2}\left[\frac{(8+\sigma)r^2h^2}{(3+\sigma)r_0^2}\log\frac{r_0}{r}\right.$$
$$\left.+\frac{5\sigma}{1-\sigma}\left\{\log\frac{r_0}{r}-1+\frac{3+\sigma}{8(1+\sigma)}\right\}z^2\right]+\cdots.$$

† In this case, we demand that $w_{00}(0)=d$, $w_{22}(0)=0$, $w_{44}(0)=0$, etc.

‡ Cf. §§ 9, 11—14 (particularly the concluding remarks of § 14).

The developments (109) meet the requirements both of convergence and differentiability; for the series (108b) converges when the magnitude of the ratio $\frac{h}{r_0}$ is suitably restricted, and is independent of z and r.

Although (99) and (106) break down at the center of the plate, they are the exact solutions under the assumption that the surface tractions are nil in the case of central load.†

11. Complete circular plate under uniform pressure and clamped at the edge. In this problem, and in that of the next paragraph, the displacements are given by formulas (69) with the terms in U_{3b} and w_{4a} included.

Let the upper base ($z = h$) be subject to the pressure p per unit of surface area. Then $S^{3'} \equiv S^3 t^3 = -p$, $S_r \equiv 0$, and equation (50) reduces to

$$w_{0b}^{\prime*\prime*} = -\frac{p}{D}; \tag{110}$$

or, after "anti-starring",

$$w_{0b}^{\prime*} = -\frac{pr}{2D} + \frac{K}{r}. \tag{111}$$

To determine the constant K, we have the relation (62b), and since the shear must be finite at the center, we find $K = 0$ and

$$N_3 t^3 = \tfrac{1}{2} pr. \tag{112}$$

If W is the total load, then $W = \pi p r_0^2$, but it is convenient to retain the p in the work that follows. Since the plate is complete, we suppress the term in $\log r$ in the solution of (111) and write

$$w_{0b} = -\frac{pr^4}{64D} + C_1 r^2 + C_2, \tag{113}$$

and when the constants of integration have been determined by means of the boundary conditions $w_{0b}(r_0) = w_{0b}'(r_0) = 0$, we find

$$w_{0b} = -\frac{p}{64D}(r_0^2 - r^2)^2. \tag{114}$$

For w_{2b}, we find an equation of the form (96b); but $K' = 0$, since the shear would otherwise become infinite at the center. Our further boundary

† Cf. Saint-Venant's *Annotated Clebsch*, Note du § 45.

conditions on w_{2b} are the same as those of § 9, and hence $w_{2b} \equiv 0$. It follows from equation (48) that

$$U_{3a} = \frac{p r \eta^2}{2(1-\sigma) D}. \tag{115}$$

The remaining unknowns of the w_{0b} system vanish identically.

It remains to determine the two unknowns of the U_{1b} system. Equations (46) and (47) reduce to

$$\sigma U_{3b} + (1-\sigma) w_{4a} = -\frac{(1-2\sigma) p \eta^3}{3(1-\sigma) D}, \tag{116a}$$

$$(1-\sigma) U_{3b}^{*\prime} + \sigma w_{4a}' = 0, \tag{116b}$$

and eliminating w_{4a} we have

$$U_{3b}^{*\prime} = 0. \tag{117}$$

Now it is clear that in addition to the condition $U(r_0, 0) = 0$ of (59), we have also $U(0,0) = 0$; but this implies that $U_{3b}(r_0) = U_{3b}(0) = 0$, and hence we find $U_{3b} \equiv 0$. It follows from (116a) that

$$w_{4a} = -\frac{(1-2\sigma) p \eta^3}{3(1-\sigma)^2 D}. \tag{118}$$

The displacements become†

$$U = -\frac{p r}{16 D}(r_0^2 - r^2) z + \frac{p r}{12(1-\sigma) D}[6h^2 z - (2-\sigma) z^3], \tag{119a}$$

$$\begin{aligned} w = &-\frac{p}{64 D}(r_0^2 - r^2)^2 + \frac{\sigma p}{16(1-\sigma) D}(r_0^2 - 2r^2) z^2 \\ &- \frac{p}{24(1-\sigma)^2 D}[8(1-2\sigma) h^3 z + 6h^2 z^2 - (1-\sigma^2) z^4]. \end{aligned} \tag{119b}$$

† These results agree with Love, p. 490.

We observe that the middle plane is bent without radial extension. The central deflection is

$$(120)\qquad d = -\frac{p r_0^4}{64 D} = -\frac{W r_0^2}{64 \pi D},$$

and by comparison with (100) is seen to be one quarter of that which would be produced by the same total load concentrated at the center.

The normal shearing force is given by (112). For the radial force and the flexural couple, we find†

$$(121)\qquad \mathrm{T} \equiv \mathrm{T}_4 t^4 = -\frac{\sigma p h}{1-\sigma},$$

$$(122)\qquad \mathrm{G} \equiv \mathrm{G}_3 t^3 + \mathrm{G}_5 t^5 = -\frac{p}{16}\left[(1+\sigma) r_0^2 - (3+\sigma) r^2\right] + \frac{(8+\sigma+\sigma^2) p h^2}{20(1-\sigma)}.$$

12. **Complete circular plate under uniform pressure and supported at the edge.** The constants in (113) are determined by the conditions $w_{0b}(r_0) = \mathrm{G}_3(r_0) = 0$, and we find

$$(123)\qquad w_{0b} = -\frac{p}{64 D}\left[\frac{2(3+\sigma)}{1+\sigma}(r_0^2 - r^2) r_0^2 - (r_0^4 - r^4)\right].$$

The determination of the unknowns w_{2b} and U_{3a} follows the lines indicated in § 10. We are led to the results

$$(124)\qquad w_{2b} = -\frac{(8+\sigma+\sigma^2) p \eta^2}{40(1-\sigma^2) D}(r_0^2 - r^2),$$

$$(125)\qquad U_{3a} = \frac{(2+9\sigma-\sigma^2) p \eta^2}{20(1-\sigma^2) D} r.$$

The remaining unknowns of the w_{0b} system vanish identically.

Turning to the unknowns U_{3b} and w_{4a}, we find the first of these functions given by (117). From $U_{3b}(0) = 0$, it follows that $U_{3b} = Kr$, and K is determined by equation (61 d), where $\mathrm{T}_4(r_0) = 0$. When U_{3b} is known, w_{4a} is given by (116 a). The results are as follows:

† Recall that p is of order t^3 and h of order t.

$$U_{3b} = \frac{\sigma p \eta^3 r}{3(1-\sigma^2)D}, \tag{126}$$

$$w_{4a} = -\frac{p\eta^3}{3(1-\sigma^2)D}. \tag{127}$$

The displacements take the form†

$$\begin{aligned} U = &-\frac{pr}{16D}\left[\frac{3+\sigma}{1+\sigma}r_0^2 - r^2\right]z \\ &\frac{pr}{60(1-\sigma^2)D}\left[20\sigma h^3 + 3(2+9\sigma-\sigma^2)h^2 z - 5(1+\sigma)(2-\sigma)z^3\right], \end{aligned} \tag{128a}$$

$$\begin{aligned} w = &-\frac{p}{64D}\left[\frac{2(3+\sigma)}{1+\sigma}(r_0^2-r^2)r_0^2 - (r_0^4-r^4)\right] \\ &-\frac{p\sigma}{8(1-\sigma)D}\left[\frac{8+\sigma+\sigma^2}{5\sigma(1+\sigma)}(r_0^2-r^2)h^2 - \left(\frac{3+\sigma}{2(1+\sigma)}r_0^2 - r^2\right)z^2\right] \\ &-\frac{p}{120(1-\sigma^2)D}\left[40h^3 z + 6(5+2\sigma+\sigma^2)h^2 z^2 - 5(1+\sigma)^2 z^4\right]. \end{aligned} \tag{128b}$$

The shearing force is given by (112), T vanishes throughout the plate, and for the flexural couple we find

$$\mathrm{G} \equiv \mathrm{G}_3 t^3 = -\frac{(3+\sigma)p}{16}(r_0^2 - r^2). \tag{129}$$

By setting $z = 0$ in (128b), we obtain the expression for the surface into which the middle plane is bent. In particular, the central deflection is

$$d = -\frac{(5+\sigma)Wr_0^2}{64(1+\sigma)\pi D}\left[1 + \frac{8(8+\sigma+\sigma^2)h^2}{5(1-\sigma)(5+\sigma)r_0^2}\right], \tag{130a}$$

where $W = \pi p r_0^2$ is the total load. The middle plane is stretched uniformly, for setting $z = 0$ in (128a) we find

$$U(r,0) = \frac{\sigma p r h^3}{3(1-\sigma^2)D} = \frac{\sigma p r}{2E}. \tag{131}$$

† Cf. Love, p. 487.

A linear element originally normal to $z = 0$ is deformed into a curved filament; the angle at which this curve cuts the surface into which the middle plane is bent is

$$\theta = \tfrac{1}{2}\pi - \lambda, \tag{132a}$$

where λ is given with sufficient accuracy by the tangent

$$\tan\lambda = \left.\frac{\dfrac{\partial U}{\partial z} + w'}{1 - \dfrac{\partial U}{\partial z}\, w'}\right|_{z=0}. \tag{132b}$$

This formulation leads to the result

$$\theta = \tfrac{1}{2}\pi - \frac{3(1+\sigma)pr}{4Eh}. \tag{133}$$

When we solve the example of this paragraph (or the preceding) from the point of view of an initially prescribed central deflection,† we are led to a situation analogous to that noted in § 10.

13. Plates of constant thickness under pressure of type such that the series for the displacements terminate. The "star" notation, besides rendering our formulas compact, enables us to see with ease the types of loading that lead to terminating series for the displacements.

Let us look first at the axial case. In the coefficient of t^{2n} in (69*b*), we observe that w_{0b} is subject n times to the pair of operations $'*$; for example,

† To solve this problem *ab initio* means that we must retain the odd S's which enter in the differential equations of the U_{1b} and w_{0b} systems. Since we seek a *uniform* loading that will cause a central deflection d, we find $w_{0b}'^{*}{}'^{*} = \text{const.}$, $w_{2b}'^{*}{}'^{*} = \text{const.}$, etc. Integrating and suppressing the log term, there are three constants to be determined. For the first equation, the boundary conditions are $w_{0b}(0) = d$, $w_{0b}(r_0) = G_3(r_0) = 0$; subsequent boundary conditions are of the type $w_{2b}(0) = w_{2b}(r_0) = G_5(r_0) = 0$, without the d. We are thus led to the result

$$-p = S^3 t^3 + S^5 t^5 + \cdots = Dw_{0b}'^{*}{}'^{*} + Dw_{2b}'^{*}{}'^{*} t^2 + \cdots$$

$$= \frac{64(1+\sigma)Dd}{(5+\sigma)r_0^4}\left[1 - \frac{8(8+\sigma+\sigma^2)h^2}{5(1-\sigma)(5+\sigma)r_0^2} + \cdots\right],$$

which is precisely what we get by solving (130*a*) for p in terms of d and expanding.

the term in t^6 contains $w_{0b}^{\prime *\prime *\prime *}$. A similar situation presents itself in (69*a*), except that here the final operation in the sequence is that of differentiation. Equation (50) reduces for the case of constant thickness to

$$w_{0b}^{\prime *\prime *} = \frac{S^3 t^3}{D}, \tag{134}$$

and it appears that the series for the displacements will terminate if and only if the pair of operations $'*$, applied a sufficient number of times to (134), causes the right-hand member of the equation to vanish. The truth of this conjecture is borne out by the structure of the differential equations of the w_{0b} system. We are assuming, of course, that the *load* is prescribed at the outset.

A little consideration shows that the only type of loading that meets the stated requirement has the form

$$\sum_{n=0}^{m} - p_{2n}\, r^{2n} + \log r \sum_{n=0}^{m} - q_{2n}\, r^{2n}. \tag{135a}$$

Here the p's and q's are constants, positive in the case of pressure. We confine our attention to pressures of the form

$$\sum_{n=0}^{m} - p_{2n}\, r^{2n}, \tag{135b}$$

for terms in the logarithm give a type of loading which not only lacks physical interest but also leads to formulas of displacement which present discontinuities at the center of the plate.

Although a load of the form

$$\sum_{n=0}^{m} - p_{2n+1}\, r^{2n+1} \tag{136}$$

fails to lead to terminating series for the displacements, it presents some features of interest. With this type of pressure, the successive processes of differentiation and starring lead at a certain stage to the starring of a constant or a constant multiplied by $\log r$. At this point, negative powers of r are introduced, and since the next operation is that of differentiation it is clear that no term can reduce to 0 at any stage. In this type of loading not only are the displacements given by infinite series, but also, due to the entrance of negative powers of r, all terms from a certain point on will present discontinuities at the center of the plate.

The phenomenon just pointed out is not without explanation. Turning to (135*b*) and (136), we are led to examine the curve $z = r^n$ with reference to

its behavior at the origin. If n is even, as in (135 b), the curve is continuous at $r = 0$, together with all its derivatives. When n is odd, as in (136), the nth derivative is discontinuous at $r = 0$ and has there a jump of amount $2\,n!$, and it is natural that such a singularity should manifest itself in the formulas of displacement.

It will be recalled that we assumed S_0^0 constant; in general it is a function of r, and if the series for the displacements are to terminate in the radial case, S_0^0 must clearly have form (135). Otherwise the successive differentiation and starring operations would at no stage lead to 0.

When $n = 0$, (135 b) yields the important case of uniform loading — and if $p_0 = 0$, central loading. The latter case is dealt with fully in §§ 9, 10. In the case of uniform loading, and for (135 b) in general, a number of interesting possibilities arise when the plate is taken to be incomplete. In point of fact, we may have

(a) both edges clamped,
(b) both edges supported,
(c) one edge clamped and one edge free,
(d) one edge clamped and one edge supported,
(e) one edge supported and one edge free.

In cases (c), (d), (e) we may interchange the rôle of inner and outer edges. Furthermore, we obtain results for a complete plate by setting the inner radius equal to 0 in any of these three cases. We are not attempting to indicate in any exhaustive manner the full scope of the applications, but it is of some interest to note the variety of problems that lead to terminating series for the displacements.

In the following paragraph, we consider an *incomplete* plate, mainly because we wish to paraphrase the problem (in § 15) for the case of variable thickness, but in this example we take the pressure to be uniform in order to simplify the computations. It may be well to have on record an example in which the load depends upon r. It is not a long piece of work to compute the following displacements for the case of a complete plate, clamped at $r = r_0$ and supporting a load $-pr^2$:

$$(137\,a)\quad U = -\frac{pr}{96D}(r_0^4 - r^4)z - \frac{pr}{4(1-\sigma)D}\Big[\frac{\sigma(r_0^2 - r^2)h^3}{3}$$

$$-\left\{r^2 + \frac{(8-3\sigma)(r_0^2 - r^2)}{10}\right\}h^2 z + \frac{(2-\sigma)r^2 z^3}{6(1-\sigma)}\Big]$$

$$+\frac{pr}{60(1-\sigma)D}[4h^4 z + 20(1-\sigma)h^3 z^2 + 40(1-3\sigma)h^2 z^3 + (3-\sigma)z^5],$$

$$
\begin{aligned}
w = & -\frac{p}{576\,D}[2\,r_0^6-3\,r_0^4\,r^4+r^6] \\
& +\frac{p}{480(1-\sigma)\,D}[3(8-3\,\sigma)(r_0^2-r^2)^2\,h^2+5\,\sigma(r_0^4-3\,r^4)\,z^2] \\
& +\frac{p}{120(1-\sigma)^2\,D}[20\{\sigma^2\,r_0^2-2\,(1-\sigma)^2\,r^2\}\,h^3\,z \\
& -3\{10\,r^2+\sigma(8-3\,\sigma)(r_0^2-2\,r^2)\}\,h^2\,z^2+5\,(1-\sigma^2)\,r^2\,z^4] \\
& -\frac{p}{180(1-\sigma)^2\,D}[6\,h^4\,z^2+40\,\sigma(1-\sigma)\,h^3\,z^3-3\,(1-\sigma)(2-3\,\sigma)h^2\,z^4 \\
& +(1-\sigma)(2+\sigma)z^6].
\end{aligned}
\tag{137b}
$$

The central deflection is

$$
d = -\frac{p\,r_0^6}{288\,D}+\frac{(8-3\,\sigma)\,p\,h^2\,r_0^4}{160\,(1-\sigma)\,D}. \tag{138}
$$

When U and w are expressed in terms of d, the coefficients, considered as functions of r and r_0, are homogeneous and of order in agreement with the rule given at the close of § 10. If U and w are written in terms of p, or in terms of the total load $\left(W=\frac{1}{4}\,\pi\,p\,r_0^4\right)$, there is still homogeneity in r and r_0, but in such cases the order is altered throughout by one and the same constant.†

14. **Incomplete circular plate under uniform pressure with one edge clamped and one edge free.** Taking $r=r_1$ as the free edge, we determine the constant in equation (111) by means of the condition $N_3(r_1)=0$ and find

$$
w_{0b}^{\prime *\prime} = -\frac{p}{2\,D\,r}(r^2-r_1^2). \tag{139}
$$

It follows that

$$
N_3(r_0)\,t^3 = \frac{p}{2\,r_0}(r_0^2-r_1^2). \tag{140}
$$

We assume that r_1 is the inner edge; if $r_1>r_0$, the parenthesis changes sign, in agreement with the conventions of § 5.

† See the concluding remarks of § 14.

Integrating (139), the three constants are determined by the conditions $G_8(r_1) = w_{0b}(r_0) = w'_{0b}(r_0) = 0$, and the final result is the following:

$$
\begin{aligned}
w_{0b} = &-\frac{p}{32QD}\Big[\{(1-\sigma)(r_0^2+2r_1^2)r_0^2+(1+3\sigma)r_1^4\}(r_0^2-r^2) \\
&+2r_0^2r_1^2\{(1+\sigma)r_0^2+(1-\sigma)(2r^2+r_1^2)\}\log\frac{r_0}{r} \\
&+4(1+\sigma)r_1^4\Big\{r_0^2\log\frac{r_0}{r_1}-r^2\log\frac{r}{r_1}\Big\}-8(1+\sigma)r_0^2r_1^4\log\frac{r_0}{r}\log\frac{r_0}{r_1}\Big] \\
&+\frac{p}{64D}[r_0^4-r^4+8(r_0^2-r^2)r_1^2],
\end{aligned} \tag{141}
$$

where the constant Q is given by

$$Q = (1-\sigma)r_0^2+(1+\sigma)r_1^2.$$

When we set $r_1 = 0$, (141) reduces to (114).

For w_{2b}, we have an equation of the form (96b), where the condition $N_5(r_1) = 0$ shows the constant must be 0. Hence we have to integrate the equation

$$w_{2b}^{(*)\prime} = 0, \tag{142}$$

and determine the three constants by means of the conditions $G_5(r_1) = w_{2b}(r_0) = w'_{2b}(r_0) = 0$. Having found w_{2b}, equation (48) gives us U_{3a}. The results are as follows:

$$w_{2b} = -\frac{(8-3\sigma)pr_1^2\eta^2}{20(1-\sigma)QD}\left[r_0^2-r^2-2r_0^2\log\frac{r_0}{r}\right]. \tag{143a}$$

$$U_{3a} = \frac{(r^2-r_1^2)p\eta^2}{2(1-\sigma)Dr}+\frac{(8-3\sigma)(r_0^2-r^2)pr_1^2\eta^2}{10(1-\sigma)QDr}. \tag{143b}$$

For U_{8b}, we have equation (117). The conditions $U_{8b}(r_0) = T_4(r_1) = 0$ give $U_{8b} \equiv 0$. It follows that w_{4a} is given by (118).

Substitution in (69) gives the following displacements:

$$
\begin{aligned}
U = &-\frac{pr}{16QD}\Bigg[(1-\sigma)(r_0^2+2r_1^2)r_0^2+(1+3\sigma)r_1^4\\
&+\{(1+\sigma)r_0^2+(1-\sigma)(2r^2+r_1^2)\}\frac{r_0^2r_1^2}{r^2}-(r^2+4r_1^2)Q\\
&-\frac{4(1+\sigma)r_0^2r_1^4}{r^2}\log\frac{r_0}{r_1}+2(1+\sigma)r_1^4\left\{2\log\frac{r}{r_1}+1\right\}\\
&-4(1-\sigma)r_0^2r_1^2\log\frac{r_0}{r}\Bigg]z\\
&+\frac{p}{12(1-\sigma)Dr}\Bigg[6\left\{r^2-r_1^2+\frac{(8-3\sigma)(r_0^2-r^2)r_1^2}{5Q}\right\}h^2z\\
&-(2-\sigma)(r^2-r_1^2)z^3\Bigg],
\end{aligned}
\tag{144a}
$$

$$
\begin{aligned}
w = &\; w_{0b} \text{ (as given above)}\\
&-\frac{p}{80(1-\sigma)QD}\Bigg[4(8-3\sigma)r_1^2\left\{r_0^2-r^2-2r_0^2\log\frac{r_0}{r}\right\}h^2\\
&-5\sigma\Bigg\{4(1+\sigma)r_1^4\left(\log\frac{r}{r_1}+1\right)-4(1-\sigma)r_0^2r_1^2\left(\log\frac{r_0}{r}-1\right)\\
&+(1-\sigma)(r_0^2+2r_1^2)r_0^2+(1+3\sigma)r_1^4-(2r^2+4r_1^2)Q\Bigg\}z^2\Bigg]\\
&-\frac{p}{120(1-\sigma)^2D}\Bigg[40(1-2\sigma)h^3z+30h^2z^2\\
&-\frac{12\sigma(8-3\sigma)r_1^2h^2z^2}{Q}-5(1-\sigma^2)z^4\Bigg].
\end{aligned}
\tag{144b}
$$

When $r_1=0$, these formulas reduce to (119). If we think of r_0 as the inner edge, we see that by setting $r_0=0$ we obtain the solution for the case of a complete plate under uniform pressure with center clamped and edge free.

In the examples of complete plates, §§ 9—13, we found the coefficients in U and w homogeneous in r and r_0; when the plate is incomplete, there is homogeneity in r, r_0 and r_1. For example, we observe in (144), where U and w

are expressed in terms of p, that the coefficient of t^n is homogeneous of order $4-n$ in r, r_0 and r_1.

The "check of dimensions" furnishes the clue to the situation in all cases. For example, when U and w are expressed in terms of d, all coefficients of d must clearly be of the 0th degree in length. We are thus led to the result that the coefficient of t^n must be of the 0th degree in length. But being homogeneous of order n in ζ and η, this coefficient is of the nth degree in length so far as z and h are concerned. It follows that the contributions made by r, r_0 and r_1 must total to degree $-n$.

The check of dimensions does not require that we write the displacements in terms of d, or that the plate be complete, or even of constant thickness. It is important to note that we have finally obtained a maximum amount of information as to the manner of occurrence not only of ζ and η, and α and its derivatives, but also of r, r_0 and r_1. It is a particular advantage to have this information available as an aid to the elimination of errors in computation.

15. **Incomplete circular plate of variable thickness under uniform pressure with one edge clamped and one edge free.** In § 8, we considered an example of variable thickness under the radial case; it remains to consider an example of variable thickness under the axial case.

Let us take $S^{\prime\prime} \equiv S^3 t^3 = -p$, $S_r \equiv 0$, where p is a constant pressure per unit of surface area and is everywhere normal to the upper base $z = a$. Under this assumption it is easy to anti-star equation (50); we find

$$(145) \qquad w_{0b}''' \alpha^3 + 3\left[w_{0b}'' + \sigma \frac{w_0'}{r}\right]\alpha^2 \alpha' = -\frac{3(1-\sigma^2)pr}{4Et^3} + \frac{K}{r}.$$

To determine K, we use (62a) where $N_3(r_1) = 0$. When this is done, we may write our equation in the form

$$(146) \qquad w_{0b}''' + w_{0b}\left[\frac{1}{r} + \frac{3a'}{a}\right] + w_{0b}'\left[\frac{3\sigma a'}{ra} - \frac{1}{r^2}\right] = -\frac{3(1-\sigma^2)p(r^2 - r_1^2)}{4Ea^3 r},$$

where $a = \alpha t$.

This equation resembles (82) in the respect that when

$$(147) \qquad \frac{a'}{a} = \frac{k}{r}$$

it is of Euler's type and readily integrated. We accordingly confine ourselves to the case that

$$(148) \qquad a(r) = hr^k.$$

and avoid difficulty at the center by taking the plate to be incomplete. Since k and h are at our disposal, (148) yields a variety of interesting shapes for the plate in question.

Substituting (148) in (146), and multiplying through by r^3, the equation to be integrated takes the form

$$(149) \qquad w'''_{0b}\, r^3 + w''_{0b}\, r^2 (1+3k) + w'_{0b}\, r\, (3\sigma k - 1) = -\frac{3(1-\sigma^2)\,p\,(r^2-r_1^2)}{4\,E h^3\, r^{3k-2}}.$$

(Applying a check of dimensions, we observe that both p and h^3 are of order t^3, and the right-hand member is of order t^0 as it should be.) This equation is the analogue of (139).

To solve (149) we set $v = \log r$. This change of independent variable gives us an equation with constant coefficients, and the homogeneous equation associated therewith has the characteristic equation

$$(150) \qquad m\,[m^2 + (3k-2)\,m - 3k(1-\sigma)] = 0.$$

Here the quadratic factor has the discriminant $\Delta = (3k-2)^2 + 12k(1-\sigma)$, which vanishes when $k = \frac{2}{3}(\sigma \pm \sqrt{\sigma^2-1})$. But since $\sigma < 1$, this value of k is never real. Thus we see that in any physical problem Δ will be positive, and equation (150) will have in general three distinct† real roots; we denote these by

$$(151) \qquad 0,\ m_1,\ m_2.$$

When we add to the solution of the homogeneous equation the particular integral‡ of the non-homogeneous equation, we find the complete solution of (149) to be the following:

---

† The only exception is when $k = 0$.

‡ The particular integral is found by the well known methods of Murray, *Introductory Course in Differential Equations*, § 70, p. 89. We exclude from the text the exceptional values $k = 0$, $k = \frac{2}{3}$, $k = \frac{4}{3}$, $k = \dfrac{8}{3(3-\sigma)}$. The case $k = 0$ is disposed of in § 14; by the methods of Murray, p. 90, the remaining special values of k lead respectively to the following particular integrals:

$$k = \frac{2}{3}:\ -\frac{3p}{16\,Eh^3}\,[(1-\sigma)\,r^2 + 2(1+\sigma)\,r_1^2 \log r];$$

$$k = \frac{4}{3}:\ \frac{3(1+\sigma)\,p}{32\,Eh^3}\left[2\log r + \frac{r_1^2}{r^2}\right];$$

$$k = \frac{8}{3(3-\sigma)}:\ -\frac{3(1+\sigma)(3-\sigma)^2 p}{32\,Eh^3}\left[\frac{r^{4(1-\sigma)/(3-\sigma)}\log r}{5-3\sigma} - \frac{r_1^2\, r^{-2(1+\sigma)/(3-\sigma)}}{2(1+\sigma)}\right].$$

$$
\tag{152}
\begin{aligned}
w_{0b} &= K_1 r^{m_1} + K_2 r^{m_2} + K_3 \\
&- \frac{3(1-\sigma^2)p}{4Eh^3}\left[\frac{r^{4-3k}}{2(4-3k)^2-3k(1-\sigma)(4-3k)} + \frac{r_1^2\, r^{2-3k}}{3k(1-\sigma)(2-3k)}\right].
\end{aligned}
$$

The three constants are determined by means of the boundary conditions $G_3(r_1) = w_{0b}(r_0) = w'_{0b}(r_0) = 0$. The result is

$$
\tag{153}
\begin{aligned}
w_{0b} &= -K_1 (r_0^{m_1} - r^{m_1}) - K_2 (r_0^{m_2} - r^{m_2}) \\
&- \frac{3(1-\sigma^2)p}{4Eh^3}\left[\frac{r^{4-3k} - r_0^{4-3k}}{2(4-3k)^2-3k(1-\sigma)(4-3k)} + \frac{r_1^2\{r^{2-3k} - r_0^{2-3k}\}}{3k(1-\sigma)(2-3k)}\right],
\end{aligned}
$$

where

$$
K_1 = \frac{3(1-\sigma^2)p}{4Eh^3 m_1}\left[\frac{A}{Q}(m_2 - 1 + \sigma)\, r_1^{m_2} + \frac{B}{Q}\, r_0^{m_2}\right],
$$

$$
K_2 = -\frac{3(1-\sigma^2)p}{4Eh^3 m_2}\left[\frac{A}{Q}(m_1 - 1 + \sigma)\, r_1^{m_1} + \frac{B}{Q}\, r_0^{m_1}\right],
$$

and the auxiliary constants are given by

$$
A = \left[\frac{r_0^2}{2(4-3k) - 3k(1-\sigma)} + \frac{r_1^2}{3k(1-\sigma)}\right] r_0^{2-3k},
$$

$$
B = \left[\frac{3k - 3 - \sigma}{2(4-3k) - 3k(1-\sigma)} + \frac{3k - 1 - \sigma}{3k(1-\sigma)}\right] r_1^{4-3k},
$$

$$
Q = (m_2 - 1 + \sigma)\, r_0^{m_1} r_1^{m_2} - (m_1 - 1 + \sigma)\, r_0^{m_2} r_1^{m_1}.
$$

We have found the leading term in the axial displacement, and we readily compute the leading term in the radial displacement. As in § 8, we find, when we proceed to the determination of further unknowns, that the differential equations are of the same type as the one just considered and hence readily integrable. The only difficulties involved are apparently in the nature of disagreeable computation.

16. **Critique of method.** In the case of plates of constant thickness, we have seen that the method of series leaves nothing to be desired with reference to rigor; in the examples of physical interest, the series for the

displacements terminate, and when the parameter t is suppressed, the formulas satisfy the equations of equilibrium identically. Furthermore, we have observed that when the central deflection is prescribed, the displacements are given by infinite series which meet the requirements both of convergence and differentiability.

When the thickness is *variable*, the formal series in t for the displacements become complicated, and as yet the author has not found opportunity to examine in any detail the convergence of these more involved developments. In the nature of the method, if we suppress the parameter t in the series ordered according to degree of homogeneity in z and h, these series are such that, when differentiated formally and substituted in the equations of equilibrium, terms of like degree annul each other; that is, the equations of equilibrium are satisfied in the usual formal sense.

Furthermore, there is a sense in which the formal series for the displacements yield approximate solutions. If we break off the series and discard terms of order higher than t^{2k+1}, we obtain formulas of displacement which correspond to a total surplus body force per unit of volume of order t^{2k} and a total surplus surface traction per unit of area of order t^{2k+1}. Thus the truncated series correspond to a total surplus applied force of order t^{2k}, whereas the total *given* applied force is at most of order t^3, that is, of the order of the cube of the ratio of the thickness of the plate to its diameter. If k is large and the ratio of thickness to diameter is small, it seems probable that when the surplus applied forces are removed, no sensible change occurs in the displacements, and that we have a physical solution of our problem. It would be of interest to check this conjecture by comparison of the potential energy in the actual and approximate solution.

We note that the present paper is concerned with the restricted theory of strain;† that is, we have suppressed the squares and products of the derivatives of U and w in writing the expressions (16) for the components of strain. Recall also that our results hold only for plates slightly bent, and that we have assumed a homogeneous and isotropic material.

Our assumption of developability in the "neighborhood of the middle plane" does not permit the satisfaction of the most general type of boundary condition at a surface $r =$ const. But if the applied tractions are distributed in the manner we have specified, our solutions are *exact*; "if they are distributed otherwise (Love, p. 480), without ceasing to be equivalent to resultants of the types T, N, G, the solution represents the state of the plate with sufficient approximation at all points which are not close to the edge".

In the usual theory, exact solutions of the more involved problems in

† Cf. Love, p. 57, on the general theory of strain.

circular plates (of constant thickness) are arrived at by arbitrary synthesis, through superposition, of several solutions of simpler type, the latter being obtained by methods which are not without complication. It appears that the method of series, when applied to plates of constant thickness, involves a minimum amount of computation and thus commends itself for the simplicity and directness of its underlying motif.

The method of series is applicable to a variety of one- and two-dimensional problems,† with thickness either uniform or variable, and its usefulness may extend to hydrodynamics, electricity, and electro-magnetism. In the problem of the circular plate we have been led to total differential equations. In general, there are partial differential equations to be dealt with, but this increase in complexity lies in the nature of the problem itself.

The method affords a solution of a problem which has interested elasticians since Poisson and Cauchy.

HARVARD UNIVERSITY,
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† The author has prepared a paper on *Rods of constant or variable circular cross section*. The method of series is immediately adaptable, and applications are made to the uniformly tapering rod, to a bulging rod, and to the right circular cylinder. This paper will appear in the American Journal of Mathematics.

As the page proof of the present paper leaves my hands I am applying the method of series to rectangular plates. In a number of cases of constant thickness the partial differential equations are tractable, and it is possible to obtain exact solutions; in such cases the problem of the *thick* plate or beam is completely solved. If we set $z = 0$ in the exact solution, there remain: i) the leading term, independent of z and h (the solution of the approximate theory), and ii) certain terms, involving h, not accounted for in the approximate theory. So far as I am aware, the only contribution to the problem of the thick rectangular plate is M. Mesnager's solution for the case of supported edges and central or uniform pressure (Comptes Rendus, vol. 164, 1917, p. 721 and vol. 165, 1917, p. 551).

PERMUTABLE RATIONAL FUNCTIONS*

BY

J. F. RITT

Introduction

We investigate, in this paper, the circumstances under which two rational functions, $\Phi(z)$ and $\Psi(z)$, each of degree greater than unity,† are such that

$$\Phi[\Psi(z)] = \Psi[\Phi(z)].$$

A pair of functions of this type will be called *permutable*.

A memoir devoted to this problem has recently been published by Julia.‡ When $\Phi(z)$ and $\Psi(z)$ are polynomials, and are such that no iterate of one is identical with any iterate of the other, Julia shows how $\Phi(z)$ and $\Psi(z)$ can be obtained from the formulas for the multiplication of the argument in the functions e^z and $\cos z$. His other results are mainly of a qualitative nature, and deal with the manner in which $\Phi(z)$ and $\Psi(z)$ behave when iterated.

Certain of Julia's results have been announced independently by Fatou.§ Fatou's method is identical with that of Julia.

The method used in the present paper differs radically from that of Julia and Fatou, and leads to results of much greater precision. Its chief yield is the

THEOREM. *If the rational functions $\Phi(z)$ and $\Psi(z)$, each of degree greater than unity, are permutable, and if no iterate of $\Phi(z)$ is identical with any iterate of $\Psi(z)$,‖ there exist a periodic meromorphic function $f(z)$, and four numbers a, b, c and d, such that*

$$f(az+b) = \Phi[f(z)], \qquad f(cz+d) = \Psi[f(z)].$$

The possibilities for $f(z)$ are: any linear function of e^z, $\cos z$, $\wp z$; in the lemniscatic case ($g_3 = 0$), $\wp^2 z$; in the equianharmonic case ($g_2 = 0$), $\wp' z$

* Presented to the Society, February 24, 1923.

† The case in which one of the functions is linear will be met incidentally in § X.

‡ *Mémoire sur la permutabilité des fractions rationnelles,* Annales de l'Ecole Normale Supérieure, vol. 39 (1922), pp. 131–215.

§ Paris Comptes Rendus, Oct. 10, 1921.

‖ The condition that no common iterate exists is certainly satisfied if the degrees of the two functions are relatively prime.

and $\wp^3 z$. These are, essentially, the only periodic meromorphic functions which have rational multiplication theorems.*

The multipliers a and c must be such that if ω is any period of $f(z)$, $a\omega$ and $c\omega$ are also periods of $f(z)$.

If p represents the order of $f(z)$, that is, the number of times $f(z)$ assumes any given value in a primitive period strip or in a primitive period parallelogram, the products

$$b(1-e^{2\pi i/p}), \qquad d(1-e^{2\pi i/p})$$

must be periods of $f(z)$.

Finally,

$$(a-1)d-(c-1)b$$

must be a period of $f(z)$.

The condition that $\Phi(z)$ and $\Psi(z)$ have no iterate in common, can be replaced by one which is certainly not stronger, and which is satisfied, for instance, if there does not exist a rational function $\sigma(z)$, of degree greater than unity, such that

$$\Phi(z) = \varphi[\sigma(z)], \qquad \Psi(z) = \psi[\sigma(z)],$$

where $\varphi(z)$ and $\psi(z)$ are rational.

The existence of the periodic function $f(z)$ is demonstrated by a method which is almost entirely algebraic. It would be interesting to know whether a proof can also be effected by the use of the Poincaré functions employed by Julia.

Of the periodic functions listed above, the linear integral functions of e^z and of $\cos z$ are the only ones whose multiplication theorems will produce a pair of permutable polynomials. In all other cases, at least one of the functions $\Phi(z)$ and $\Psi(z)$ will be fractional. In § X we settle completely the case in which $\Phi(z)$ and $\Psi(z)$ are both polynomials, obtaining the

THEOREM. *If $\Phi(z)$ and $\Psi(z)$ are a pair of permutable polynomials (non-linear), which do not come from the multiplication theorems of e^z and* $\cos z$, *there exist a linear integral function $\lambda(z)$ and a polynomial*

$$G(z) = zR(z^r),$$

* These Transactions, vol. 23 (1922), p. 16.

where $R(z)$ is a polynomial, such that

$$\Phi(z) = \lambda^{-1}\{\varepsilon_1 G^{(\mu)}[\lambda(z)]\}, \qquad \Psi(z) = \lambda^{-1}\{\varepsilon_2 G^{(\nu)}[\lambda(z)]\},$$

where $G^{(i)}(z)$ represents the ith iterate of $G(z)$, and where ε_1 and ε_2 are rth roots of unity.

Thus, neglecting a linear transformation, $\Phi(z)$ and $\Psi(z)$ are iterates of the same polynomial, multiplied sometimes by roots of unity.

As to the permutable fractional functions which do not come from the multiplication theorems (and which therefore have an iterate in common), we give a method for constructing them, which, while it is not everything to be desired, still throws considerable light upon the functions under consideration. This method, which involves two types of operations, applies to all rational functions, integral or fractional.

Let $\Phi(z)$ and $\Psi(z)$ be two permutable rational functions. If there exist three rational functions, $\varphi(z)$, $\psi(z)$ and $\sigma(z)$, each of degree greater than unity, such that

$$\Phi(z) = \sigma[\varphi(z)], \qquad \Psi(z) = \sigma[\psi(z)],$$

and that $\varphi[\sigma(z)]$ is permutable with $\psi[\sigma(z)]$, we shall call the act of passing from $\Phi(z)$ and $\Psi(z)$ to $\varphi[\sigma(z)]$ and $\psi[\sigma(z)]$ an *operation of the first type.*

If $\Phi(z)$ and $\Psi(z)$ are permutable, it is evident that $\Phi[\Psi(z)]$ will be permutable both with $\Phi(z)$ and with $\Psi(z)$. We shall call the act of passing from $\Phi(z)$ and $\Psi(z)$ to $\Phi[\Psi(z)]$ and $\Phi(z)$, or to $\Phi[\Psi(z)]$ and $\Psi(z)$, an *operation of the second type.*

We show in § X that *if $\Phi(z)$ and $\Psi(z)$ do not come from the multiplication theorems of the periodic functions, there exists a linear function $\lambda(z)$ such that $\lambda^{-1}\Phi\lambda(z)$ and $\lambda^{-1}\Psi\lambda(z)$ can be obtained by repeated operations of the above two types, starting from a pair of functions*

$$zR(z^r), \qquad \varepsilon z R(z^r),$$

where $R(z)$ is a rational function, and where ε is an rth root of unity (sometimes unity itself).

For polynomials, only operations of the second type are necessary, and we obtain the explicit formulas given above. In the case of the fractional functions, however, operations of the first type are sometimes necessary, so that there exist permutable pairs of fractional functions which come neither from

the multiplication theorems of the periodic functions, nor from the iteration of a function.

We have not succeeded thus far in determining all cases in which operations of the first type are possible. In fact, an illustration of the operations of that type, given in § X, will probably weaken any a priori conviction one might have to the effect that formulas as explicit as those stated above for polynomials can be found for the permutable fractional functions which have an iterate in common. It will not be inconceivable that too little order may prevail among the functions of that class for a complete enumeration of them to be possible.

The present paper is the outcome of efforts to solve, for fractional functions, the problem settled for polynomials in our paper *Prime and composite polynomials.**

I. Preliminaries

What we do principally in this section is to recall certain results proved in the above mentioned paper on prime and composite polynomials, on which the work in the present paper will be based.

Let $\varphi(z)$ and $\psi(z)$ be two rational functions of the respective degrees $r>1$ and $s>1$. Let $F(z)=\varphi[\psi(z)]$. We put

$$w = F(z) = \varphi(u), \qquad u = \psi(z).$$

It is easy to see the Riemann surface for $F^{-1}(w)$ is related to those for $\varphi^{-1}(w)$ and $\psi^{-1}(u)$. Suppose that, for $u=c$, $\psi^{-1}(u)$ has a critical point with a certain number of cycles. As $\varphi^{-1}(w)$ assumes no value more than once on its Riemann surface, $F^{-1}(w)$ will surely have a critical point for $w=\varphi(c)$. If the value c is assumed by a branch u_1 of $\varphi^{-1}(w)$ which is uniform in the neighborhood of $\varphi(c)$, those branches of $F^{-1}(w)$ for which $\psi(z)=u_1$ will be ramified at $\varphi(c)$ as the branches of $\psi^{-1}(u)$ are at c. If the value c is assumed by a cycle of p branches of $\varphi^{-1}(w)$, each cycle of $\psi^{-1}(u)$ at $u=c$ will lead to a cycle of $F^{-1}(w)$ at $\varphi(c)$ with p times as many sheets. If $\varphi^{-1}(w)$ has a critical point for $w=d$, and if none of the points $u=\varphi^{-1}(d)$ is a critical point of $\psi^{-1}(u)$, then each cycle of $\varphi^{-1}(w)$ at d leads to s cycles of the same number of sheets for $F^{-1}(w)$ at d.

We call the sum of the orders of the branch points of an algebraic function which are superimposed on each other at a given critical point the index of the function at the point. The sum of the indices of the inverse of a rational function of degree r, for all of its critical points, is $2r-2$.

* These Transactions, vol. 23 (1922), p. 51.

It is easy to see that the index of $F^{-1}(w)$, at any critical point w_0 of $\varphi^{-1}(w)$, is at least s times the index of $\varphi^{-1}(w)$ at w_0. Also, if the index of $F^{-1}(w)$ at w_0 is q, and if w_0 is not a critical point of $\varphi^{-1}(w)$, then $\psi^{-1}(u)$ must have critical points whose affixes are values of $\varphi^{-1}(w)$ at w_0, and the sum of whose indices is q.

With respect to the group of monodromy of $F^{-1}(w)$, the rs branches of $F(w)$ break up into r systems of imprimitivity, such that if the branches

$$z_1, z_2, \ldots, z_s \tag{1}$$

constitute one of these systems, we have

$$\psi(z_1) = \psi(z_2) = \cdots = \psi(z_s).\text{*} \tag{2}$$

These r systems are said to be *determined* by $\psi(z)$.

Conversely, let $F(z)$ be any rational function of degree rs whose inverse has an imprimitive group. If (1) is a system of imprimitivity of the group of $F^{-1}(w)$, there exists a rational function $\psi(z)$, of degree s, for which (2) holds, and we have

$$F(z) = \varphi[\psi(z)],$$

where $\varphi(z)$ is a rational function of degree r. If another rational function determines the same systems of imprimitivity as $\psi(z)$, it is a linear function of $\psi(z)$.

Suppose that (1) can be broken up into smaller systems of imprimitivity, each containing t letters, and let $\sigma(z)$ be the rational function of degree t which determines these systems. Then $\psi(z)$ is a rational function of $\sigma(z)$.†

We shall deal next with five rational functions of degrees greater than unity; $\varphi_1(z)$ and $\varphi_2(z)$, each of degree r, $\psi_1(z)$ and $\psi_2(z)$, each of degree s, and $F(z)$. We suppose that

$$F(z) = \varphi_1[\psi_2(z)] = \psi_1[\varphi_2(z)].$$

We put $w = F(z)$, $u = \psi_2(z)$ and $v = \varphi_2(z)$, so that

$$w = \varphi_1(u) = \psi_1(v).$$

* Loc. cit., p. 54.

† Loc. cit., p. 55.

The function $\psi_2(z)$ determines r systems of imprimitivity of the group of $F^{-1}(w)$,

$$U_1, U_2, \ldots, U_r,$$

each containing s letters, while $\varphi_2(z)$ determines the s systems

$$V_1, V_2, \ldots, V_s,$$

each containing r letters. If w describes a closed path, the sets U are permuted like the branches of $\varphi_1^{-1}(w)$, and the sets V like the branches of $\psi_1^{-1}(w)$.

In what follows, we shall assume that each system U has exactly one letter in common with each system V. We see directly that if some substitution of the group of $F^{-1}(w)$ interchanges the letters of some set V_i among themselves, it interchanges the sets U with a substitution similar to that which it effects on the letters of V_i.

A point (w, u) on the Riemann surface of $\varphi_1^{-1}(w)$ for which w is a critical point of $\varphi_1^{-1}(w)$ or of $\psi_1^{-1}(w)$ (that is of $F^{-1}(w)$) will be called a *point* of $\varphi_1^{-1}(w)$. A point of $\psi_1^{-1}(w)$ is defined similarly. If p branches coalesce at a point, we shall say that the point is of *order* p. Thus a point of order p is a branch point of order $p-1$.

A point of order unity will be called a *simple point*. If $\varphi_1^{-1}(w)$ has a point of order p for $w = w_0$, and if $\psi_1^{-1}(w)$ has a critical point at w_0 where its branches undergo a substitution whose order is not a factor of p, the point of order p will be called an *A-point* of $\varphi_1^{-1}(w)$. An A-point of $\psi_1^{-1}(w)$ is defined similarly. If w_0 is a critical point of $\psi_1^{-1}(w)$, every simple point which $\varphi_1^{-1}(w)$ may have at w_0 is an A-point of $\varphi_1^{-1}(w)$.

Suppose that $\psi_1^{-1}(w)$ has a point of order p for $w = w_0$, and let the value of $\psi_1^{-1}(w)$ at the point be v_0. Suppose that as w makes a turn about w_0, the branches of $\varphi_1^{-1}(w)$ undergo a substitution S; S will be identity if w_0 is not a critical point of $\varphi_1^{-1}(w)$. Let w execute p turns about w_0, so that the branches of $\varphi_1^{-1}(w)$, and therefore the systems U, undergo the substitution S^p. Let v_i be one of those branches of $\psi_1^{-1}(w)$ which coalesce at the point of order p now under consideration. As w makes p turns about w_0, the value of v_i makes a single turn about v_0. Thus, by the p turns, the letters of V_i are interchanged among themselves; hence the substitution which these letters undergo is similar to S^p. This means that when v makes a turn about v_0, the branches of $\varphi_2^{-1}(v)$ undergo a substitution similar to S^p.

Thus, *a necessary and sufficient condition that $\varphi_2^{-1}(v)$ have a critical point at v_0, is that the value v_0 be assumed by $\psi_1^{-1}(w)$ at an A-point.* A similar result holds for $\psi_2^{-1}(u)$.

II. THE THREE SEQUENCES

We deal with the two permutable functions $\Phi(z)$ and $\Psi(z)$, of the respective degrees $m>1$ and $n>1$, and write

$$w = F(z) = \Phi[\Psi(z)] = \Psi[\Phi(z)],$$

or, more briefly,

$$F = \Phi\Psi = \Psi\Phi.$$

The greatest single source of work in this paper is the possibility of the existence of a rational function $\sigma_0(z)$, of degree greater than unity, such that

$$\Phi = \varphi_0\sigma_0, \qquad \Psi = \psi_0\sigma_0,$$

where $\varphi_0(z)$ and $\psi_0(z)$ are rational (even linear). Wherever the contrary is not stated, we shall assume that such a $\sigma_0(z)$ exists.

With a view towards securing later a sharp separation of permutable pairs of functions into two classes, we establish now a definite method for selecting $\sigma_0(z)$. We proceed as follows. As $F = \Phi\Psi$, the function $\Psi(z)$ determines m systems of imprimitivity of the group of $F^{-1}(w)$, each containing n letters. Also, if $\Psi = \psi_0\sigma_0$, the function $\sigma_0(z)$ determines systems of imprimitivity of the group of $F^{-1}(w)$. Two branches, z_1 and z_2, will be in the same one of the systems determined by $\sigma_0(z)$ if $\sigma_0(z_1) = \sigma_0(z_2)$. But as $\Psi(z_1) = \Psi(z_2)$ in this case, z_1 and z_2 are both in one system determined by $\Psi(z)$. Hence every system determined by $\sigma_0(z)$ is contained in a system determined by $\Psi(z)$, so that each system determined by $\Psi(z)$ is composed of one or more systems determined by $\sigma_0(z)$. A similar fact is true of $\Phi(z)$.

Thus, given any system determined by $\Phi(z)$, there is at least one system determined by $\Psi(z)$ with which it has more than one letter in common. Now, it is a simple consequence of the elementary notions on imprimitivity that if a group has two sets of systems of imprimitivity, there exists a number t_0, such that if a system of the first set has at least one letter in common with a system of the second set, it has precisely t_0 letters in common with it. These systems of t_0 letters are themselves systems of imprimitivity.

We shall suppose, in what follows, that $\sigma_0(z)$ is so taken that it determines the systems of imprimitivity of t_0 letters just shown to exist. This determines

$\sigma_0(z)$ to within a linear function, and the particular disposition made in regard to this linear function is of no importance for what follows.

We see now that there exists no rational function $\beta(z)$, of degree greater than unity, such that

$$\varphi_0 = \zeta\beta, \qquad \psi_0 = \xi\beta,$$

where $\zeta(z)$ and $\xi(z)$ are rational. If there did, each system of the group of $F^{-1}(w)$ determined by $\Phi(z)$ would have more than t_0 letters in common with some system determined by $\Psi(z)$.

We have

$$\varphi_0\,\sigma_0\,\psi_0\,\sigma_0 = \psi_0\,\sigma_0\,\varphi_0\,\sigma_0.$$

so that

(3) $$\varphi_0\,\sigma_0\,\psi_0 = \psi_0\,\sigma_0\,\varphi_0.$$

Let the degrees of $\varphi_0(z)$, $\psi_0(z)$, $\sigma_0(z)$ be r_0, s_0, t_0, respectively. We represent each member of (3) by G. The function $\sigma_0\,\psi_0$ determines r_0 systems of imprimitivity of the group of G^{-1},* each containing $s_0\,t_0$ letters. Also, $\sigma_0\,\varphi_0$ determines s_0 systems of imprimitivity of the group of G^{-1}, each containing $r_0\,t_0$ letters. The $s_0\,t_0$ letters in any system determined by $\sigma_0\,\psi_0$ are distributed among the s_0 systems determined by $\sigma_0\,\varphi_0$. Hence given any system determined by $\sigma_0\,\psi_0$, there is some system determined by $\sigma_0\,\varphi_0$ with which it has at least t_0 letters in common. Thus, by what goes before, there exists a number $t_1 \geqq t_0$, such that if a system determined by $\sigma_0\,\psi_0$ has at least one letter in common with a system determined by $\sigma_0\,\varphi_0$, the two systems have precisely t_1 letters in common. The systems determined by $\sigma_0\,\psi_0$ and by $\sigma_0\,\varphi_0$ break up into a third set of systems of imprimitivity, which are determined by a function $\sigma_1(z)$, of degree t_1. Also, we have

$$\sigma_0\,\varphi_0 = \varphi_1\,\sigma_1, \qquad \sigma_0\,\psi_0 = \psi_1\,\sigma_1,$$

where $\varphi_1(z)$ and $\psi_1(z)$ are rational functions of the respective degrees $r_1 \leqq r_0$ and $s_1 \leqq s_0$. Furthermore, there exists no rational $\beta(z)$ of degree greater than unity such that

$$\varphi_1 = \zeta\beta, \qquad \psi_1 = \xi\beta,$$

where $\zeta(z)$ and $\xi(z)$ are rational. Finally, by (3),

$$\varphi_0\,\psi_1\,\sigma_0 = \psi_0\,\varphi_1\,\sigma_0,$$

* If φ_0 is linear, we have to consider all of the branches of G^{-1} as forming a single system of imprimitivity.

so that

$$\varphi_0 \psi_1 = \psi_0 \varphi_1 .$$

We now subject the permutable functions $\varphi_1 \sigma_1$ and $\psi_1 \sigma_1$* to the treatment given above to $\Phi(z)$ and $\Psi(z)$. What follows is plain. There exist three sequences

$$(A) \qquad \varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_i, \ldots$$

$$(B) \qquad \psi_0, \psi_1, \psi_2, \ldots, \psi_i, \ldots$$

$$(C) \qquad \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_i, \ldots$$

the degrees of the functions in the first two sequences being non-increasing, those of the functions in the third sequence non-decreasing, and the following relations holding for every i:

$$\varphi_i \sigma_i \psi_i \sigma_i = \psi_i \sigma_i \varphi_i \sigma_i,$$

$$\sigma_i \varphi_i = \varphi_{i+1} \sigma_{i+1}, \qquad \sigma_i \psi_i = \psi_{i+1} \sigma_{i+1},$$

$$\varphi_i \psi_{i+1} = \psi_i \varphi_{i+1};$$

furthermore, for no i does a function $\beta(z)$ of degree greater than unity exist such that

$$\varphi_i = \zeta\beta, \qquad \psi_i = \xi\beta,$$

where $\zeta(z)$ and $\xi(z)$ are rational.

For the case in which no $\sigma_0(z)$ exists, we define the sequences (A) and (B) by the equations

$$\varphi_i = \Phi, \qquad \psi_i = \Psi,$$

and all facts proved for (A) and (B) when (C) exists will hold also in this case.

From the monotonic character of the degrees of the functions in the three sequences, and from the fact that the degree of every $\sigma_i(z)$ is not greater than the smaller of m and n, we derive the

CONCLUSION. *There exist a subscript i_0, and three integers r, s and t, such that, for $i \geqq i_0$, $\varphi_i(z)$, $\psi_i(z)$ and $\sigma_i(z)$ are of the respective degrees r, s and t.*

* We have from (3), $\sigma_0 \varphi_0 \sigma_0 \psi_0 = \sigma_0 \psi_0 \sigma_0 \varphi_0$.

From this point on, until we come to the last section of our paper, we shall work under the assumption that r and s each exceed unity. We assume, that is, that for no i is one of the permutable functions $\varphi_i\,\sigma_i$ and $\psi_i\,\sigma_i$ a rational function of the other. This understood, we prove the important

LEMMA. *There exist a subscript i_1, and two integers $h \leqq 4$ and $k \leqq 4$, such that, for $i \geqq i_1$, every $\varphi_i(z)$ is of degree r, while its inverse has precisely h critical points; and every $\psi_i(z)$ is of degree s, while its inverse has precisely k critical points.*

We write

$$w = F(z) = \varphi_i\,[\psi_{i+1}(z)] = \psi_i\,[\varphi_{i+1}(z)],$$

and assume that $i \geqq i_0$, so that $\varphi_i(z)$ and $\varphi_{i+1}(z)$ are each of degree r, and $\psi_i(z)$ and $\psi_{i+1}(z)$ each of degree s. As there exists no non-linear rational $\beta(z)$ such that $\varphi_{i+1} = \zeta\beta$, $\psi_{i+1} = \xi\beta$, where $\zeta(z)$ and $\xi(z)$ are rational, each of the systems of imprimitivity of the group of $F^{-1}(w)$ determined by $\varphi_{i+1}(z)$ has precisely one letter in common with each system determined by $\psi_{i+1}(z)$. Hence we can employ the notion of the A-point.

Suppose that φ_i^{-1} has g critical points, $w_1, \ldots, w_g$. We seek a lower bound for the sum of the indices of ψ_i^{-1} at these g points. That sum equals $gs - j$, where j is the number of points which ψ_i^{-1} has at $w_1, \ldots, w_g$. If p of these points are simple points, the other $j-p$ are at least of order 2, and we have

$$2(j-p)+p \leqq gs, \tag{4}$$

so that $j \leqq (gs+p)/2$, and the sum of the indices of ψ_i^{-1} at $w_1, \ldots, w_g$ is at least $(gs-p)/2$. We observe that if one of the j points is of order greater than 2, or if there are fewer than p simple points, (4) must be an inequality, and the sum of the indices of ψ_i^{-1} at $w_1, \ldots, w_g$ will exceed $(gs-p)/2$.

Suppose now that φ_{i+1}^{-1} has fewer than g critical points. Then ψ_i^{-1} must have fewer than g simple points at $w_1, \ldots, w_g$, for every such simple point of ψ_i^{-1} is an A-point, and yields a critical point of φ_{i+1}^{-1}. Hence the sum of the indices of ψ_i^{-1} at $w_1, \ldots, w_g$ exceeds $(gs-g)/2$ and we have

$$\frac{gs-g}{2} < 2s-2,$$

so that $g < 4$. Thus φ_i^{-1} has three critical points, and φ_{i+1}^{-1} has two.

At each of the two critical points of φ_{i+1}^{-1}, its r branches must be permuted in a single cycle. From the manner in which the critical points of φ_{i+2}^{-1} depend

on those of φ_{i+1}^{-1}, we see that, at every critical point of φ_{i+2}^{-1}, its branches undergo a substitution which is a power of a cyclic substitution in r letters. Such a substitution must be regular, that is, it displaces every letter, and the order of its cycles are all equal. It follows that, for $j > i$, the critical points of every φ_j^{-1} have regular substitutions. Hence at every critical point of $\varphi_j^{-1} (j > i)$, all of the branches of φ_j^{-1} are permuted, so that the index of φ_j^{-1}, at each of its critical points, is at least $r/2$. As the sum of the indices of every φ_j^{-1} is $2r - 2$, every φ_j^{-1} has either two critical points or three.

If a $\varphi_j^{-1} (j > i)$ has three critical points, it cannot have one at which its branches are permuted in a single cycle; in that case the sum of its indices would exceed $2r - 2$. Hence φ_{j+1}^{-1} must also have three critical points. for φ_j^{-1} cannot transmit to φ_{j+1}^{-1} a critical point with a single cycle.

We see now that if φ_{i+1}^{-1} has fewer critical points than φ_i^{-1}, then, either each φ_j^{-1} has two critical points for $j > i$, or else, for j sufficiently large, each φ_j^{-1} has three critical points. It remains only to settle the case in which, for every $i > i_0$, φ_{i+1}^{-1} has at least as many critical points as φ_i^{-1}. In this case, since φ_i^{-1} cannot have more than $2r - 2$ critical points, is is evident that an h exists, such that, for i sufficiently large, each φ_i^{-1} has precisely h critical points. When φ_i^{-1} and φ_{i+1}^{-1} have an equal number of critical points, we find that the g of the preceding page does not exceed 4. Hence $h \leqq 4$.

The argument for $\varphi_i(z)$ holds also for $\psi_i(z)$, and the lemma is proved.

III. The critical points of φ_i^{-1} and ψ_i^{-1}

From this point on, every subscript i will be understood to be not less than the i_1 of the preceding lemma. We assume also that $h \geqq k$; if this is not so at the start, we need only interchange the designations of $\Phi(z)$ and $\Psi(z)$. Let $h = 4$, and let the critical points of φ_i^{-1} be $w_1, \ldots, w_4$. If ψ_i^{-1} has $p \leqq 4$ simple points at $w_1, \ldots, w_4$, the sum of the indices of ψ_i^{-1} at $w_1, \ldots, w_4$ is at least $(4s - p)/2$; if one of the points of ψ_i^{-1} at $w_1, \ldots, w_4$ is of order greater than 2, this lower bound must be increased. We must thus have

$$\frac{4s - p}{2} \leqq 2s - 2. \tag{5}$$

If p were less than 4, or if ψ_i^{-1} had a point of order greater than 2 at $w_1, \ldots, w_4$ (5) could not hold. Hence $p = 4$, and those points of ψ_i^{-1} at $w_1, \ldots, w_4$ which are not simple are all of order 2. Furthermore, it is clear that ψ_i^{-1} can have no critical points other than $w_1, \ldots, w_4$.

We shall see below that the points of φ_i^{-1} are also all of order 2, except four which are simple.

Let $h=3$, and let the critical points of φ_i^{-1} be a, b, and c. We show first that ψ_i^{-1} has no critical points other than a, b, c.

Suppose first that $s>3$. If a were not a critical point of ψ_i^{-1}, ψ_i^{-1} would have at least four simple points at a, so that φ_{i+1}^{-1} would have at least four critical points. Hence a, and similarly b and c, are critical points of ψ_i^{-1}. As $k\leqq 3$, ψ_{i+1}^{-1} can have no other critical points.

Let $s=3$. If a is not a critical point of ψ_i^{-1}, ψ_i^{-1} must have no simple point at b or at c. This means that the branches of ψ_i^{-1} are permuted in a single cycle at b and at c, so that b and c are the only critical points of ψ_i^{-1}.

Let $s=2$; ψ_i^{-1} has just two branch points, which are both simple. If one or both of these were not placed at a, b, c, at least two of these latter points would not be critical points of ψ_i^{-1}, and ψ_i^{-1} would have at least four A-points.

Thus, when $h=3$, ψ_i^{-1} has no critical points other than a, b, c. When $h=3$, we must have $r\geqq 3$. But we have seen above that when $h=3$ and $k=2$, we have $s\leqq 3$. Hence, in the case of $h=3$, it is legitimate to assume that $r\geqq s$. In §§ V, VII, VIII, IX, which deal with the case of $h=3$, it will be understood, unless otherwise stated, that $r\geqq s$.

Suppose that $h=3$, and that at the critical points a, b, c, of φ_i^{-1}, the branches of φ_i^{-1} undergo substitutions of orders α_i, β_i, γ_i, respectively. The function ψ_i^{-1} must have precisely three A-points. Let x be the sum of the orders of the A-points of ψ_i^{-1} at a, y at b and z at c. The orders of those points of ψ_i^{-1} at a which are not A-points are divisible by α_i. Hence their number is at most $(s-x)/\alpha_i$. Thus, the total number of points of ψ_i^{-1} at a, b, c is at most

$$\frac{s-x}{\alpha_i}+\frac{s-y}{\beta_i}+\frac{s-z}{\gamma_i}+3. \tag{6}$$

As ψ_i^{-1} has no critical points other than a, b, c, the sum of its indices at a, b, c is $2s-2$. This means that ψ_i^{-1} has $s+2$ points at a, b, c. Thus the expression (6) is at least $s+2$, and we find, directly,

$$\frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}\geqq 1-\frac{1}{s}+\frac{1}{s}\left(\frac{x}{\alpha_i}+\frac{y}{\beta_i}+\frac{z}{\gamma_i}\right). \tag{7}$$

In particular, the first member of (7) exceeds $(1-1/s)$. Suppose that ψ_i^{-1} has at a an A-point of order g. This A-point gives rise to a critical point of φ_{i+1}^{-1} at which the branches of φ_{i+1}^{-1} undergo a substitution similar to the gth power of the substitution which the branches of φ_i^{-1} undergo at a. The order, call it α_{i+1}, of the substitution at this critical point of φ_{i+1}^{-1} is α_i divided by

the greatest common divisor of α_i and g. Certainly, then, $1/\alpha_{i+1}$ is not greater than g/α_i. It is easy now to see that

$$\frac{x}{\alpha_i}+\frac{y}{\beta_i}+\frac{z}{\gamma_i}\geqq\frac{1}{\alpha_{i+1}}+\frac{1}{\beta_{i+1}}+\frac{1}{\gamma_{i+1}},$$

so that

$$(8)\qquad \frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}\geqq 1-\frac{1}{s}+\frac{1}{s}\left(\frac{1}{\alpha_{i+1}}+\frac{1}{\beta_{i+1}}+\frac{1}{\gamma_{i+1}}\right).$$

But the quantity in parentheses in the second member of (8) exceeds $1-1/s$, the lower bound secured above for the first member. Hence

$$\frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}>1-\frac{1}{s}+\frac{1}{s}\left(1-\frac{1}{s}\right)=1-\frac{1}{s^2}.$$

This gives a new lower bound for the quantity in parentheses in (8), which, when substituted, shows that the first member is not less than $1-1/s^3$. As this process may be repeated indefinitely, we arrive at the

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$$(9)\qquad \frac{1}{\alpha_i}+\frac{1}{\beta_i}+\frac{1}{\gamma_i}\geqq 1.$$

When $h=4$, the same procedure leads easily to the result that the sum of the reciprocals of the orders of the substitutions at the four critical points of φ_i^{-1} is at least 2. As each order is at least 2, each order must be precisely 2. Thus, all points of φ_i^{-1} which are not simple are of order 2. It follows, as at the beginning of this section, that φ_i^{-1} has precisely four simple points.

We suppose finally that $h=2$. Two cases have to be considered:

(*a*) Either r or s exceeds 2. In this case, φ_i^{-1} and ψ_i^{-1} must have the same two critical points, else one of them would have more than two simple points, and these would be A-points. At both critical points, the branches of both functions are permuted in single cycles.

(*b*) $r=s=2$. In this case, φ_i^{-1} and ψ_i^{-1} cannot have the same two critical points, else neither would have an A-point. If they had no critical point in common, each would have four A-points. Hence we may assume that φ_i^{-1} has simple branch points at each of two points a and b, and that ψ_i^{-1} has one simple branch point at a, and one at a third point c.

In what follows, special attention has to be given to certain cases in which s is small. A device which would permit us to assume that r and s are arbitrarily large, one based, for instance, on replacing the permutable functions by iterates of themselves, would eliminate many painful paragraphs.

IV. THE MULTIPLICATION FORMULAS FOR e^z; THE POWERS OF z

We consider the case in which $h = 2$, and in which r and s are not both 2. In this case, φ_i^{-1} and ψ_i^{-1} each have the same two critical points, a_i and b_i. Both branch points of either function are A-points of that function. Also, the values a_{i+1} and b_{i+1} which φ_i^{-1} assumes at its A-points are the same two values which ψ_i^{-1} assumes at its A-points, for these are the affixes of the critical points of φ_{i+1}^{-1} and ψ_{i+1}^{-1}.*

Turning now to the sequence (C), we shall prove that σ_i^{-1} has no critical points other than a_{i+1} and b_{i+1}. Suppose that σ_i^{-1} has such critical points other than a_{i+1} and b_{i+1}, and let m_i be the sum of its indices at those additional critical points. As $m_i \leqq 2t-2$, there is an i for which m_i is a maximum. Suppose that it is this i with which we are dealing. Consider the relation

$$\sigma_i \varphi_i = \varphi_{i+1} \sigma_{i+1}. \tag{10}$$

From a remark made at the beginning of § I, we see that the sum of the indices of $\varphi_i^{-1} \sigma_i^{-1}$ at the critical points of σ_i^{-1} other than a_{i+1} and b_{i+1} is at least rm_i. Turning to the second member of (10) we see that σ_{i+1}^{-1} has critical points whose affixes are not values of φ_{i+1}^{-1} at a_{i+1} and b_{i+1}, that is, not a_{i+2} or b_{i+2}, at which the sum of its indices is at least rm_i. This contradicts the assumption that m_i is a maximum.

Thus the inverses of $\varphi_i \sigma_i$ and $\psi_i \sigma_i$ have the two critical points a_i, b_i, at which all of their branches are permuted in single cycles. From this, and from the relation

$$\sigma_i \psi_i = \psi_{i+1} \sigma_{i+1},$$

it follows that the two critical points of ψ_i^{-1} are the values which σ_i^{-1} assumes at a_{i+1} and b_{i+1}. Hence the inverse of

$$F = \varphi_i \sigma_i \psi_i \sigma_i$$

* It is not necessary that φ_i^{-1} and ψ_i^{-1} should assume the same value at a particular A-point.

has only the two critical points a_i and b_i. Also, from the condition of permutability, we see that the two values a_i and b_i are assumed by the inverses of both $\varphi_i\,\sigma_i$ and $\psi_i\,\sigma_i$ at a_i and b_i. These same values are assumed by F^{-1} at a_i and b_i.

The relation

$$\varphi_i\,\sigma_i\,\psi_i\,\sigma_i = \sigma_{i-1}\,\varphi_{i-1}\,\sigma_{i-1}\,\psi_{i-1}$$

shows that σ_{i-1} has no critical point other than a_i and b_i, and that the inverse of $\varphi_{i-1}\,\sigma_{i-1}\,\psi_{i-1}$ has no critical points other than a_{i-1} and b_{i-1}, the values (so we designate them) which σ_{i-1}^{-1} assumes at its critical points. Hence the inverse of the function

$$\varphi_{i-1}\,\sigma_{i-1}\,\psi_{i-1}\,\sigma_{i-1} = \psi_{i-1}\,\sigma_{i-1}\,\varphi_{i-1}\,\sigma_{i-1}$$

has no critical points other than a_{i-1} and b_{i-1}.

Thus the inverses of $\varphi_{i-1}\,\sigma_{i-1}$ and $\psi_{i-1}\,\sigma_{i-1}$ have a_{i-1} and b_{i-1} as their only critical points, and the values assumed by the inverses at their critical points are a_{i-1} and b_{i-1}.

Continuing thus, we find that the inverses of the original permutable functions

$$\Phi = \varphi_0\,\sigma_0, \qquad \Psi = \psi_0\,\sigma_0$$

have two critical points a_0 and b_0, and that the values of Φ^{-1} and Ψ^{-1} at their critical points are a_0 and b_0.

The statement just made in regard to $\Phi(z)$ and $\Psi(z)$ holds also if no σ_0 exists; it is only necessary to discard those parts of the proof which involve a σ.

Let $\lambda(z)$ be any linear function such that $\lambda(a_0) = 0$, and $\lambda(b_0) = \infty$. Let

$$\Phi_1 = \lambda\,\Phi\,\lambda^{-1}, \qquad \Psi_1 = \lambda\,\Psi\,\lambda^{-1}.$$

Then Φ_1^{-1} and Ψ_1^{-1} have the two critical points 0 and ∞, and their values at these points are 0 and ∞.

It follows that

$$\Phi_1(z) = \eta\,z^p, \qquad \Psi_1(z) = \varepsilon\,z^q,$$

where $p = \pm m$, and $q = \pm n$. If we multiply $\lambda(z)$ by a suitable constant, we will have $\eta = 1$. The condition of permutability then gives $\varepsilon^{p-1} = 1$.

Thus all pairs of permutable functions, of the type considered in this section, are found by transforming with a linear function the permutable pair z^p and εz^q where p and q are positive or negative integers, and where $\varepsilon^{p-1} = 1$.

V. THE MULTIPLICATION FORMULAS FOR $\cos z$

This section and §§ VII, VIII, IX will handle the cases in which $h = 3$. We saw in § III that it is permissible to assume, when $h = 3$, that $r \geqq s$; where the contrary is not stated, this assumption will be understood to hold.

In all cases for which $h = 3$, we represent the critical points of φ_i^{-1} by a_i, b_i, c_i, and the orders of the substitutions which the branches of φ_i^{-1} undergo at a_i, b_i, c_i by $\alpha_i, \beta_i, \gamma_i$, respectively. The orders of the substitutions which the branches of ψ_i^{-1} undergo at a_i, b_i, c_i we denote by $\alpha_i', \beta_i', \gamma_i'$, respectively.

We consider in this section the case in which, for some $i \geqq i_1$, two of the orders $\alpha_i, \beta_i, \gamma_i$, say α_i and β_i, equal 2. The i used in this section is supposed to be of the type just described, and stays fixed throughout our work.

Consider first the case in which $k = 3$. We know that the critical points of ψ_i^{-1} are a_i, b_i, c_i. We are going to show that $\alpha_i' = \beta_i' = 2$.

As $r \geqq 3$, and as the branch points of φ_i^{-1} at a_i and b_i are all simple, φ_i^{-1} has at least four points (together) at a_i and b_i. If α_i' and β_i' both exceeded 2, all of these points would be A-points, whereas we know that φ_i^{-1} has only three A-points. We may suppose thus that $\alpha_i' = 2$.

If $r > 6$, φ_i^{-1} has at least four points at b_i, so that β_i' is also 2. The cases in which $r \leqq 6$, which create a type of nuisance of which there will be more later, we have to examine in detail.

Suppose that $\beta_i' > 2$. If $r = 6$, φ_i^{-1} must have three branch points (simple), at b_i; if it had fewer, it would have four or more A-points at b_i. Hence, $\alpha_{i+1}', \beta_{i+1}', \gamma_{i+1}'$, the orders of the substitutions at the critical points of ψ_{i+1}^{-1}, are all equal. Thus, by (9), $3/\alpha_{i+1} \geqq 1$, and α_{i+1} is either 3 or 2.

Suppose that $\alpha_{i+1}' = 3$. As the branch points of φ_i^{-1} at b_i are all simple, the substitutions at the critical points of ψ_{i+1}^{-1} are all similar to the square of the substitution which the branches of ψ_i^{-1} undergo at b_i, so that $\beta_i' = 6$, or $\beta_i' = 3$. If $\beta_i' = 6$, we see, remembering that $r \geqq s$, that $s = 6$. This produces the absurdity that the sum of the indices of ψ_{i+1}^{-1} is 12, or more than $2s - 2$. If $\beta_i' = 3$, we have $s \geqq 3$. If $s > 3$, then, since β_i is prime to β_i', ψ_i^{-1} has at least two A-points at b_i, so that φ_{i+1}^{-1} must have at least two critical points at which its branch points are simple. This contradicts the fact that $\alpha_{i+1}', \beta_{i+1}', \gamma_{i+1}'$ all equal 3, for we saw above that one of them must be 2 if two of $\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}$ are 2. Finally, if $s = 3$, ψ_i^{-1} has an A-point at a_i, as well as at b_i, so that the argument just used applies.

Suppose that $\alpha_{i+1}' = 2$. The index of ψ_{i+1}^{-1} cannot exceed $s/2$ at any critical point. Hence,

$$\frac{3s}{2} \geqq 2s - 2.$$

or $s \leqq 4$. We cannot have $s = 3$, as ψ_{i+1}^{-1} would have only three simple branch points in that case. If $s = 4$, there must be two simple branch points at each critical point of ψ_{i+1}^{-1}. Hence ψ_{i+1}^{-1} would have an even number of A-points, whereas it must have three. This completes the proof that $\beta_i' = 2$ when $r = 6$.

When $r = 5$, φ_i^{-1} has at least one A-point at a_i. If $\beta_i' > 2$. it would have at least three at b. Thus $\beta_i' = 2$.

Suppose that $r = 4$ and that $\beta_i' = 2$. We must have $s \leqq 4$, so that $\beta_i' \leqq 4$. If φ_i^{-1} has two simple branch points both at a_i and at b_i, we may suppose that the substitutions at these points are (12) (34) and (13) (24) respectively. Hence the substitution at c_i is (14) (23). This, as seen above, leads to the absurdity that φ_i^{-1} has an even number of A-points.

Thus φ_i^{-1} must have two simple points, either at a_i or at b_i. As φ_i^{-1} has at least two A-points at b_i when $\beta_i' > 2$, the simple points cannot be at a_i, else φ_i^{-1} would have at least four A-points. Then φ_i^{-1} has a simple branch point and two simple points at b_i. Suppose that $s = 4$. If β_i' were 4, ψ_{i+1}^{-1} would have two critical points of index 3, and a third critical point, which is impossible, since the sum of its indices is 6. If β_i' were 3, we would have $\alpha_{i+1}' = \beta_{i+1}' = \gamma_{i+1}' = 3$, and also φ_{i+1}^{-1} would have two critical points with only simple branch points, which come from the two A-points of ψ_i^{-1} at b_i. This was proved impossible above. If $s = 3$, and $\beta_i' = 3$, ψ_{i+1}^{-1} would have three critical points of index 2, an impossibility.

Suppose that $r = 3$ and that $\beta_i' > 2$. We find the absurdity that ψ_{i+1}^{-1} has two critical points of index 2 which come from b_i, and a third critical point which comes from a_i.

We have proved that $\alpha_i' = \beta_i' = 2$.

We shall now show that φ_i^{-1} and ψ_i^{-1} each have two simple points (two in all), at a_i and b_i. Consider φ_i^{-1} for instance. If r is odd, the two simple points certainly exist. In the case where r is even, if there were no such simple points, the sum of the indices of φ_i^{-1} at a_i and b_i would be r. Hence the index at c_i would be $r - 2$, and φ_i^{-1} would have precisely two points at c_i. These would be the only possible A-points of φ_i^{-1}, whereas there have to be three.

It follows from the above that, at c_i, all of the branches of φ_i^{-1} are permuted in a single cycle. The same is true of ψ_i^{-1}.

We examine now the case in which $h = 3$, $\alpha_i = \beta_i = 2$, and $k = 2$.

As seen in § III, s must be 3 or 2. If s were 3, the two branch points of ψ_i^{-1} would be points of order 3. At least one of them would be situated either at a_i or at b_i, and would be an A-point of ψ_i^{-1}. Also ψ_i^{-1} would have 3 A-points at that critical point of φ_i^{-1} which is not a critical point of ψ_i^{-1}.

Thus $s = 2$. If a_i and b_i were both critical points of ψ_i^{-1}, ψ_i^{-1} would have only two A-points. Hence we may assume that the critical points of ψ_i^{-1} are b_i and c_i.

We shall prove that r is odd, from which it will follow that φ_i^{-1} has one simple point at a_i, one at b_i, and that its branches are permuted in a single cycle at c_i.

Suppose, contrarily, that r is even. We know that φ_i^{-1} cannot have more than two simple points (in all) at a_i and b_i; if it did, its index at c_i would exceed $r-1$. Then φ_i^{-1} can have no simple point at a_i. If it had one, it would have two, and as ψ_i^{-1} has two A-points at a_i, φ_{i+1}^{-1} would have two critical points with simple branch points and with four simple points, a condition as impossible for φ_{i+1}^{-1} as for φ_i^{-1}.

It follows that φ_{i+1}^{-1} has two critical points with simple branch points and no simple points. Furthermore, it is permissible to replace $i+1$ by i, and to assume that φ_i^{-1} has no simple point at a_i or at b_i.

This understood, it follows that φ_i^{-1} has just two points at c_i, which must both be A-points. Consider the relation

$$F = \varphi_i\, \psi_{i+1} = \psi_i\, \varphi_{i+1}. \tag{11}$$

From $F = \varphi_i\, \psi_{i+1}$, we see that the two values which F^{-1} assumes at c_i are the two values which ψ_{i+1}^{-1} assumes at its critical points. As only one branch point of ψ_{i+1}^{-1} is an A-point, only one of the values of F^{-1} at c_i can be an affix of a critical point of φ_{i+2}^{-1}, or, a fortiori, of ψ_{i+2}^{-1}. Now φ_{i+1}^{-1} has two critical points with simple branch points and no simple points, at the points whose affixes are the values of ψ_i^{-1} at a_i. Hence at the point whose affix is the value of ψ_i^{-1} at c_i (c_i is an A-point of ψ_i^{-1}), φ_{i+1}^{-1} has two points, which, as in the case of φ_i^{-1}, must be A-points. This, since $F = \psi_i\, \varphi_{i+1}$, entails the contradiction that the values of F^{-1} at c_i are both affixes of critical points of ψ_{i+2}^{-1}. Thus r is odd.

When $h = k = 3$, the values of φ_i^{-1} and ψ_i^{-1} at their simple points at a_i and b_i must be the same, namely, the affixes of the critical points with simple branch points of φ_{i+1}^{-1} and ψ_{i+1}^{-1}. Similarly, φ_i^{-1} and ψ_i^{-1} must have the same single value at c_i.

Suppose that, when $k = 2$, $\varphi_i^{-1}(c_i) = c_{i+1}$ and $\psi_i^{-1}(c_i) = c'_{i+1}$. We shall show that $c_{i+1} = c'_{i+1}$.

We know that ψ_{i+1}^{-1} has a critical point at c_{i+1}, and that φ_{i+1}^{-1} has a critical point at c'_{i+1} at which its sheets are permuted in a single cycle. Let c'_{i+2} and c_{i+2} be the values of ψ_{i+1}^{-1} and φ_{i+1}^{-1} at c_{i+1} and c'_{i+1} respectively. From (11), since F^{-1} has only one value at c_i, we have $c_{i+2} = c'_{i+2}$. But since φ_{i+1}^{-1} has an A-point where its branches are permuted in a single cycle, c_{i+2} is a critical point of ψ_{i+2}^{-1}, and hence of φ_{i+2}^{-1}. Thus ψ_{i+1}^{-1} must have an A-point at its critical point at c_{i+1}. But the only branch point of ψ_{i+1}^{-1} which is an

A-point is the one at which the branches of φ_{i+1}^{-1} are permuted in a single cycle. Hence $c_{i+1} = c'_{i+1}$.

When $k = 2$, the value of φ_i^{-1} at its simple point at b_i is the affix of that critical point of ψ_{i+1}^{-1} at which the branches of φ_{i+1}^{-1} are permuted in pairs. We shall show that the value of φ_i^{-1} at its simple point at a_i is the affix of the second critical point of φ_{i+1}^{-1} where its branches are permuted in pairs.

As, by (11), $F = \psi_i \varphi_{i+1}$, we see that F^{-1} has precisely two uniform branches at a_i, whose values are the values of φ_{i+1}^{-1} at its simple points. One of these values is the affix of a critical point of φ_{i+2}^{-1} at which the branches of φ_{i+2}^{-1} are permuted in pairs. From $F = \varphi_i \psi_{i+1}$ we see now that at the point whose affix is the value of φ_i^{-1} at its simple point at a_i, both branches of ψ_{i+1}^{-1} are uniform, and the value of at least one of them is the affix of a critical point of φ_{i+2}^{-1}. This can be so only if the value of φ_i^{-1} at its simple point at a_i is the affix of a critical point of φ_{i+1}^{-1} where the branches of φ_{i+1}^{-1} are permuted in pairs.

From what precedes, we see that when $h = 3$, and $\alpha_i = \beta_i = 2$, both φ_i^{-1} and ψ_i^{-1} have two simple points at a_i and b_i, and the values of φ_i^{-1} and ψ_i^{-1} at their simple points are the same. We shall call these values a_{i+1} and b_{i+1}. Also, at c_i, φ_i^{-1} and ψ_i^{-1} both have their branches permuted in a single cycle, and assume a common single value, which we shall call c_{i+1}.

It is hardly necessary to mention the fact that for every j greater than the i used in our work above, φ_j^{-1} and ψ_j^{-1} will have the properties proved for φ_i^{-1} and ψ_i^{-1}.

Precisely as in the preceding section, we can show that σ_i^{-1} has no critical points other than a_{i+1}, b_{i+1} and c_{i+1}.

We shall prove now that all branch points which σ_i^{-1} has at a_{i+1} and b_{i+1} are simple.

Suppose that for some $j \geqq i$, some of the branch points of σ_j^{-1} at a_{j+1} and b_{j+1} are not simple, and let m_j be the sum of the orders of the branch points which are not simple. Since $m_j \leqq 2t-2$, there is a j for which m_j is a maximum. We deal with such a j.

Consider the relation

$$\sigma_j \varphi_j = \varphi_{j+1} \sigma_{j+1}. \tag{12}$$

We see that the inverse of $\sigma_j \varphi_j$ has at a_{j+1} and b_{j+1} branch points which are not simple, and the sum of whose orders is at least rm_j. As the critical points of σ_{j+1}^{-1} which are values of φ_{j+1}^{-1} at a_{j+1} and b_{j+1} are a_{j+2} and b_{j+2}, the values of φ_{j+1}^{-1} at its simple points, and as the branch points of φ_{j+1}^{-1} at a_{j+1} and b_{j+1} are all simple, we see that σ_{j+1}^{-1} has branch points at a_{j+2} and b_{j+2} which are not simple, and the sum of whose orders is at least rm_j. This contradicts the assumption that m_j is a maximum.

Thus, the inverse of $\varphi_i \sigma_i$ has no critical point other than a_i, b_i, c_i, and at a_i and b_i its branch points are all simple.

If every branch of σ_i^{-1} were permuted at a_{i+1} and at b_{i+1}, σ_i^{-1} would have just two distinct values at c_{i+1}, for its index at c_{i+1} would be $t-2$. Thus φ_i^{-1}, since it has three critical points, would have at least one critical point which is a value assumed by σ_i^{-1} at some point other than c_{i+1}. This means that the inverse of $\sigma_i \varphi_i$ would either have more critical points than a_{i+1}, b_{i+1}, c_{i+1}. or else it would have branch points of order greater than unity at a_{i+1} or b_{i+1}. This is impossible by (12), according to what we know of the critical points of the second member of (12). Hence there are, at a_{i+1} and b_{i+1}, precisely two places on the Riemann surface of σ_i^{-1} at which σ_i^{-1} is uniform. By (12), the value of σ_i^{-1} at these are a_i and b_i. Also, at c_{i+1}, the branches of σ_i^{-1} are permuted in a single cycle, and $\sigma_i^{-1}(c_{i+1}) = c_i$.

Without difficulty, we see now that the inverse of

$$F = \varphi_i \sigma_i \psi_i \sigma_i$$

has the two critical points a_i and b_i at which its branches are permuted in pairs, and the critical point c_i at which its branches are permuted in a single cycle. Also, at a_i and b_i, there are two places on the surface of F^{-1} at which F^{-1} is uniform, and the values of F^{-1} at these places are a_i and b_i. Finally, $F^{-1}(c_i) = c_i$.

From the relation

$$\varphi_i \sigma_i \psi_i \sigma_i = \sigma_{i-1} \varphi_{i-1} \sigma_{i-1} \psi_{i-1}.$$

we see that σ_{i-1}^{-1} has no critical points other than a_i, b_i, c_i, that at a_i and b_i its branch points are all simple, and that at c_i its branches are permuted in a single cycle. Also, at a_i and b_i there are two places on the surface of σ_{i-1}^{-1} at which σ_{i-1}^{-1} is uniform, assuming certain values a_{i-1} and b_{i-1}.

We let $\sigma_{i-1}^{-1}(c_i) = c_{i-1}$. It is easy to see that the inverse of

$$\varphi_{i-1} \sigma_{i-1} \psi_{i-1}$$

has a critical point at c_{i-1} at which its branches are permuted in a single cycle. At a_{i-1} and b_{i-1} its branches are permuted in pairs, except that there are two places where the inverse is uniform and takes the values a_i and b_i. Then the inverse of

$$\varphi_{i-1} \sigma_{i-1} \psi_{i-1} \sigma_{i-1}$$

has all its branches permuted in a single cycle at c_{i-1}; and all its branches permuted in pairs at a_{i-1} and b_{i-1}, except that there are two places where the inverse is uniform and takes the values a_{i-1} and b_{i-1}. The value at c_{i-1} is c_{i-1}.

Continuing thus, we prove that the inverse of

$$\Phi\Psi = \Psi\Phi$$

has three critical points a_0, b_0, c_0, where it behaves in the manner already frequently described.

Also, Φ^{-1} and Ψ^{-1} have the critical points a_0, b_0, c_0. It is obvious that $\Phi^{-1}(c_0) = \Psi^{-1}(c_0) = c_0$. Furthermore, as Φ and Ψ are at least of degree 4, their inverses both actually have critical points at a_0 and b_0, so that the values which each inverse assumes at those places at a_0 and b_0 where it is uniform are a_0 and b_0.

When the sequence (C) does not exist, we may take $i = 0$, so that $\varphi_0 = \varphi_1 = \Phi$, $\psi_0 = \psi_1 = \Psi$. The values a_1, b_1, c_1, which Φ^{-1} and Ψ^{-1} assume at their simple points and at c_0, are seen directly to be the same as a_0, b_0, c_0.

Let $\lambda(z)$ be a linear function such that

$$\lambda(a_0) = 1, \qquad \lambda(b_0) = -1, \qquad \lambda(c_0) = \infty.$$

Then the inverses of the two permutable functions

$$\Phi_1 = \lambda\Phi\lambda^{-1}, \qquad \Psi_1 = \lambda\Psi\lambda^{-1}$$

have simple branch points at 1 and -1, and their branches are permuted in a single cycle at ∞. The values of their uniform branches at 1 and -1 are 1 and -1, and $\Phi_1^{-1}(\infty) = \Psi_1^{-1}(\infty) = \infty$.

Consider the function $\cos z$. Wherever it assumes either of the values 1 or -1, it assumes it twice. It never assumes the value ∞. Hence, if we operate on $\cos z$ with Φ_1^{-1}, the n values of $\Phi_1^{-1}(\cos z)$ are uniform *im kleinen*, and therefore, also, uniform *im großen*. As Φ_1^{-1} assumes the value ∞ only at ∞, these n functions are entire.

Let $f(z)$ be one of these entire functions. As $\Phi^{-1}(\cos z)$ assumes a value 1 or -1 only when $\cos z$ is 1 or -1, and then only through a uniform branch of Φ_1^{-1}, $f(z)$ cannot assume a value 1 or -1 unless it assumes it twice. Also, $f(z)$ is never infinite.

Consider the function arc cos z. Its only finite critical points are 1 and -1, at which its branches are permuted in pairs. Hence, by the same reasoning used above, there are an infinite number of entire functions arc cos $f(z)$. Let $\beta(z)$ be one of these. We shall show that $\beta(z)$ is linear.

Wherever $\Phi_1^{-1}(z)$ is large, its modulus is of the order of $\sqrt[m]{|z|}$, where m is the degree of Φ_1. Now, since

$$|\cos z| \leqq e^{|z|},$$

there is a k such that

$$|f(z)| < k\, e^{|z|^m},$$

for every z.

Now if

$$\beta(z) = u(x, y) + i v(x, y)$$

were not linear, there would be values of z of large modulus for which

$$v(x, y) < -|z|^2,$$

and hence for which

$$|f(z)| = |\cos \beta(z)| \geqq \frac{e^{|z|^2} - e^{-|z|^2}}{2}.$$

This contradicts the first inequality for $|f(z)|$.

Thus there is a relation

$$\cos z = \Phi_1 [\cos (pz + q)],$$

or, what is the same, a relation

$$\cos (az + b) = \Phi_1 (\cos z). \tag{13}$$

Similarly, we have

$$\cos (cz + d) = \Psi_1 (\cos z).$$

As the first member of (13) has the primitive period $2\pi/a$, and as the second member has a period 2π, we see that a is an integer. In fact, we must have $a = \pm m$, where m is the degree of Φ_1. To determine b, we note that the first member of (13) must be, like the second, an even function of z. Hence, when

$$z_1 + z_2 = 2\pi,$$

we must have

$$az_1 + az_2 + 2b = 2k\pi,$$

where k is some integer. It follows that $2b$ is a multiple of 2π, so that b is either 0 or π (neglecting multiples of 2π). Similarly, $c = \pm n$, and d is either 0 or π.

Finally, since

$$\cos(acz + ad + b) = \Phi_1 \Psi_1(\cos z),$$

$$\cos(acz + cb + d) = \Psi_1 \Phi_1(\cos z),$$

we must have

$$(a-1)d \equiv (c-1)b \pmod{2\pi}.$$

As in the preceding section, all permutable pairs of functions of the type now considered can be found by transforming $\Phi_1(z)$ and $\Psi_1(z)$ with a linear function.

VI. The multiplication formulas for $\wp z$

We are going to settle, in this section, the following two cases:

(*a*) $h = 4$,

(*b*) $h = k = 2$ and $r = s = 2$. (See next to last paragraph of § III.)

Let us examine Case (*a*). We must have $k = 4$, 3 or 2.

Suppose that $k = 4$. Then, by § III, φ_i^{-1} and ψ_i^{-1} must have the same four critical points, at which each has, in all, four simple points. Furthermore, the four values which φ_i^{-1} assumes at its simple points are the same that ψ_i^{-1} assumes at its simple points, for these four values are the affixes of the common critical points of φ_{i+1}^{-1} and ψ_{i+1}^{-1}.

If $k = 3$, the degree s of ψ_i^{-1} must be 4, and ψ_i^{-1} must have three critical points at which its branches are permuted in pairs. The four simple points of ψ_i^{-1} are at a critical point w_0 of φ_i^{-1}. As the four critical points of φ_{i+1}^{-1} will all have the same index that φ_i^{-1} has at w_0, that index must be $(r-1)/2$. Hence r is odd, and φ_i^{-1} has one simple point at each of its four critical points.

Suppose that $k = 2$. We must have $s = 2$. Two of the critical points of φ_i^{-1} are critical points of ψ_i^{-1}. The other two, call them w_1 and w_2, are not.

We shall show that the four values which φ_i^{-1} assumes at its simple points are the affixes of the four critical points of φ_{i+1}^{-1}, that is, the four values of ψ_i^{-1} at w_1 and w_2.

Since φ_{i+1}^{-1} has two critical points of the type that φ_i^{-1} has at w_1, and two of the type that φ_i^{-1} has at w_1, it follows that φ_i^{-1} has (in all) two simple

points at w_1 and w_2. The other two simple points of φ_i^{-1} are A-points of φ_i^{-1}, and the values which φ_i^{-1} takes at them are critical points of φ_{i+1}^{-1}. We write

$$F = \varphi_i\,\psi_{i+1} = \psi_i\,\varphi_{i+1}.$$

From the relation $F = \psi_i\,\varphi_{i+1}$, we see that, at w_1 and w_2, there are precisely four places on the Riemann surface of F^{-1} at which F^{-1} is uniform, and that, at these four places, the values of F^{-1} are the values which φ_{i+1}^{-1} assumes at its four simple points. Let u_1 and u_2 be the values of φ_i^{-1} at its simple points at w_1 and w_2. From what we have just seen, and from the relation $F = \varphi_i\,\psi_{i+1}$, it follows that neither u_1 nor u_2 is a critical point of ψ_{i+1}^{-1}, and that the values of ψ_{i+1}^{-1} at u_1 and u_2 are the four values of φ_{i+1}^{-1} at its simple points. As at least two of these values are critical points of φ_{i+2}^{-1}, it follows that at least one of the points u_1 and u_2 is a critical point of φ_{i+1}^{-1}. We have proved that at least three of the values of φ_i^{-1} at its simple points are critical points of φ_{i+1}^{-1}. This implies, of course, that at least three of the values of φ_{i+1}^{-1} at its simple points are critical points of φ_{i+2}^{-1}. Going back three sentences, we see that u_1 and u_2 are both critical points of φ_{i+1}^{-1}, as was to be proved.

By similar reasoning, only more briefly, it can be shown that, when $k = 3$, φ_i^{-1} and ψ_i^{-1} assume the same four values at their simple points.

We shall now examine Case (b), and show that it may be amalgamated with Case (a).

Let a_i and b_i be the critical points of φ_i^{-1}, and a_i and c_i those of ψ_i^{-1}. The inverse of

$$F = \varphi_i\,\psi_{i+1} = \psi_i\,\varphi_{i+1}$$

has the three critical points a_i, b_i, c_i, at which its branches are permuted in pairs.

As φ_{i+1}^{-1} and ψ_{i+1}^{-1} have precisely one critical point in common, one, and only one, of the values of φ_i^{-1} at c_i equals a value of ψ_i^{-1} at b_i. Let the values of φ_i^{-1} at c_i be a_{i+1} and c_{i+1}, and the values of ψ_i^{-1} at b_i be a_{i+1} and b_{i+1}. There is a point d_i at which φ_i^{-1} has the value b_{i+1}. Evidently d_i is distinct from c_i. We shall show that d_i is distinct from a_i and from b_i.

As a_{i+1} and b_{i+1} are the critical points of φ_{i+1}^{-1}, and a_{i+1} and c_{i+1} those of ψ_{i+1}^{-1}, one and only one value of φ_{i+1}^{-1} at c_{i+1} equals a value of ψ_{i+1}^{-1} at b_{i+1}. Let the values of φ_{i+1}^{-1} at c_{i+1} be a_{i+2} and c_{i+2}, and those of ψ_{i+1}^{-1} at b_{i+1} be a_{i+2} and b_{i+2}. From $F = \varphi_i\,\psi_{i+1}$, we see that where φ_i^{-1} takes the value b_{i+1}, F^{-1} has branches with values a_{i+2} and b_{i+2}. But from $F = \psi_i\,\varphi_{i+1}$, we see that at the same point at which one branch of F^{-1} has the value a_{i+2},

another has the value c_{i+2}. Hence where F^{-1} takes the value a_{i+2}, it has at least three values, and therefore four. Thus F^{-1} cannot assume the value a_{i+2} at either a_i or b_i, as it has only two distinct values at these points. This proves that d_i is distinct from a_i and c_i.

Let b_{i+1} and d_{i+1} be the two values of φ_i^{-1} at d_i. We shall prove that the values of ψ_i^{-1} at d_i are c_{i+1} and d_{i+1}.

The relation $F = \varphi_i \psi_{i+1}$ shows that two of the values of F^{-1} at d_i are a_{i+2} and b_{i+2}. It follows from $F = \psi_i \varphi_{i+1}$ that ψ_i^{-1} assumes the value c_{i+1} at d_i.

Thus, since $F = \psi_i \varphi_{i+1}$, three of the values of F^{-1} at d_i are $a_{i+2}, b_{i+2}, c_{i+2}$. Hence, from $F = \varphi_i \psi_{i+1}$, we see that ψ_{i+1}^{-1} assumes the value c_{i+2} at d_{i+1}. But the proof above that φ_i^{-1} assumes the value b_{i+1} at the same point at which ψ_i^{-1} assumes the value c_{i+1} proves also that if ψ_{i+1}^{-1} assumes the value c_{i+2} at d_{i+1}, φ_{i+1}^{-1} must assume the value b_{i+2} at d_{i+1}. Hence, from $F = \psi_i \varphi_{i+1}$, we find that ψ_i^{-1} assumes the value d_{i+1} at d_i.

Returning to Case (a), we denote the critical points of φ_i^{-1} and ψ_i^{-1}, for every $i \geqq i_1$, by a_i, b_i, c_i, d_i. This will permit us to treat Cases (a) and (b) together.

The functions σ are introduced as in the two preceding cases. It is seen immediately that σ_i^{-1} has no critical points other than $a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}$, and that its branch points are all simple. This shows that the inverses of $\varphi_i \sigma_i$ and $\psi_i \sigma_i$ have no critical points other than a_i, etc., that their branch points are all simple, and that there are exactly four places on the Riemann surfaces of the inverses at a_i, etc., where the inverses are uniform.

When $h = 4$, the relation

$$\sigma_i \varphi_i = \varphi_{i+1} \sigma_{i+1} \tag{14}$$

shows that the values which σ_i^{-1} assumes where it is uniform at a_{i+1}, etc., are a_i etc. We shall show that the same holds when $h = 2$. First, the above relation shows that two of these values of σ_i^{-1} are a_i and b_i. Similarly, the relation $\sigma_i \psi_i = \psi_{i+1} \sigma_{i+1}$ shows that two of the values are a_i and c_i. It remains to show that the fourth value is d_i. If it were not, then since the values of φ_i^{-1} at d_i are b_{i+1} and d_{i+1}, the inverse of $\sigma_i \varphi_i$ would take the values b_{i+1} and d_{i+1} at some point distinct from a_i, etc. But the argument just given for σ_i^{-1} shows that σ_{i+1}^{-1} assumes the values $a_{i+1}, b_{i+1}, c_{i+1}$, at a_{i+2} etc., which means that the inverse of $\varphi_{i+1} \sigma_{i+1}$ assumes those three values at a_{i+1} etc. This, with (14), yields a contradiction.

It is visible now that the inverse of

$$F = \varphi_i \sigma_i \psi_i \sigma_i$$

has the four critical points a_i, etc., that its branch points are all simple, that there are just four places on the surface of F^{-1} at a_i, etc., where F^{-1} is uniform, and that the values of F^{-1} at these places are a_i, etc.

We now work back to Φ and Ψ. From the relation

$$F = \varphi_i \sigma_i \psi_i \sigma_i = \sigma_{i-1} \varphi_{i-1} \sigma_{i-1} \psi_{i-1}, \tag{15}$$

we see that σ_{i-1}^{-1} has no critical points other than a_i, etc., that its branch points are all simple, and that there are, at a_i, etc., just four places on the surface of σ_{i-1}^{-1} at which σ_{i-1}^{-1} is uniform. Let the values of σ_{i-1}^{-1} at these four places be a_{i-1}, etc. We see by (15) that the inverse of $\varphi_{i-1} \sigma_{i-1} \psi_{i-1}$ has the critical points a_{i-1}, etc. (all four, because it has at least eight branches), where its branches are permuted in pairs, except that there are four places at a_{i-1}, etc., where the inverse is uniform and takes the values a_i, etc. Then the inverse of

$$\varphi_{i-1} \sigma_{i-1} \psi_{i-1} \sigma_{i-1}$$

has the four critical points a_{i-1}, etc., with simple branch points, and has four places at a_{i-1}, etc., where it is uniform and takes the values a_{i-1}, etc.

Proceeding thus, we find that the inverse of

$$\Phi \Psi = \Psi \Phi \tag{16}$$

has four critical points a_0, b_0, c_0, d_0, at which its branches are permuted in pairs, except that there are four places where the inverse is uniform, assuming the values a_0, etc.

As to Φ^{-1} and Ψ^{-1}, we see that they have no critical points other than a_0, etc., and that their branch points are all simple. When Φ^{-1} and Ψ^{-1} each have four critical points, it is evident that their values where they are uniform at a_0, etc., are a_0, etc.

If $m > 4$, Φ^{-1} will have four critical points, so that the values of Ψ where it is uniform at a_0, etc., are a_0, etc. Suppose that $m > 4$, that $n = 4$, and that Ψ^{-1} has only three critical points. It is clear that at three of the places at a_0, etc., at which Φ^{-1} is uniform, it assumes values from among a_0, etc. If it did not assume one of these values at the fourth place, we would have the contradiction that the first member of (16) has four uniform branches at one of the points a_0, etc., which do not assume the values a_0, etc.

When $m = 4$, we must also have $n = 4$, so that $\varphi_i, \psi_i, \sigma_i$, are all of degree 2, and we may assume, above, that $i = 0$. It was shown above that

in this case the values of the inverses of $\varphi_i \sigma_i$ and $\psi_i \sigma_i$, where they are uniform at a_i, etc., are a_i etc.

Suppose that the sequence (C) does not exist. We may take $i = 0$. As a_1, etc., play the same rôle in regard to φ_1 and ψ_1 as a_0, etc., do in regard to φ_0 and ψ_0, and as $\varphi_0 = \varphi_1 = \Phi$, $\psi_0 = \psi_1 = \Psi$, we see immediately that the values which Φ^{-1} and Ψ^{-1} take at their uniform places at a_0, etc., are a_0, etc.

Let $\lambda(z)$ be a linear function such that $\lambda(a_0) = \infty$ and that

$$\lambda(b_0) + \lambda(c_0) + \lambda(d_0) = 0.$$

It does not matter, in this, which point is called a_0. We put

$$e_1 = \lambda(b_0), \qquad e_2 = \lambda(c_0), \qquad e_3 = \lambda(d_0).$$

We consider the two functions

$$\Phi_1 = \lambda \Phi \lambda^{-1}, \qquad \Psi_1 = \lambda \Psi \lambda^{-1},$$

whose inverses have only simple branch points, which are found at e_1, e_2, e_3, ∞.

Construct the elliptic function $\wp z$ such that $\wp(\omega_i) = e_i$, $i = 1, 2, 3$. This is possible because $e_1 + e_2 + e_3 = 0$. Furthermore, the orientation of the numbers e_i is of no importance.

As $\wp z$ assumes the values e_i and ∞ twice wherever it assumes one of them, and as the branch points of Φ_1^{-1} are all simple, the n values of $\Phi_1^{-1}(\wp z)$ are uniform *im kleinen*, and hence also *im großen*. Let $f(z)$ represent one of the n meromorphic functions $\Phi_1^{-1}(\wp z)$. As Φ_1^{-1} assumes a value e_i or ∞ only where it is uniform, $f(z)$ cannot assume a value e_i or ∞ unless it assumes it twice.

Let $\wp^{-1} z$ be the inverse of $\wp z$. Then $\wp^{-1} z$ has the four critical points e_i and ∞, and its branch points are all simple. Hence there are an infinity of meromorphic functions $\wp^{-1}[f(z)]$. Let $\beta(z)$ be one of these. Then

$$f(z) = \wp[\beta(z)],$$

or

(17) $$\wp z = \Phi_1 \wp[\beta(z)].$$

Now $\wp[\beta(z)]$, being algebraically related to $\wp z$, is an elliptic function. Then $\beta(z)$ must be entire, for if it had a pole with a finite affix $\wp[\beta(z)]$ would have an essential singularity with a finite affix.

Differentiating (17), we find

$$\wp'(z) = \Phi_1' \wp[\beta(z)]\, \wp'[\beta(z)]\, \beta'(z). \tag{18}$$

Now $\wp'[\beta(z)]$, being algebraically related to $\wp[\beta(z)]$, is an elliptic function. It follows from (18) that $\beta'(z)$ is an elliptic function. As $\beta'(z)$ is entire, it must be a constant. Thus $\beta(z)$ is linear.

There exists thus a relation

$$\wp(z) = \Phi_1[\wp(pz+q)],$$

or, what is the same, a relation

$$\wp(az+b) = \Phi_1(\wp z). \tag{19}$$

Similarly, there is a relation

$$\wp(cz+d) = \Psi_1(\wp z). \tag{20}$$

It remains only to characterize the constants. From (19) we see first, since $2\omega_1$ and $2\omega_3$ are periods of the first member, that

$$2a\omega_1 \equiv 0, \qquad 2a\omega_3 \equiv 0 \qquad (\text{modd}\, 2\omega_1, 2\omega_3). \tag{21}$$

Also, since the first member of (19) has to be an even function of z, like the second, we find

$$2b \equiv 0 \qquad (\text{modd}\, 2\omega_1, 2\omega_3). \tag{22}$$

From (20), we find, similarly,

$$2c\omega_1 \equiv 0, \qquad 2c\omega_3 \equiv 0, \qquad 2b \equiv 0 \qquad (\text{modd}\, 2\omega_1, 2\omega_3). \tag{23}$$

Finally, from the condition of permutability, we find

$$(a-1)d \equiv (c-1)b \qquad (\text{modd}\, 2\omega_1, 2\omega_3). \tag{24}$$

Any set of constants which satisfy (21), (22), (23), (24) yield a pair of permutable functions. All pairs of permutable functions, of the type considered in this section, are found by transforming Φ_1 and Ψ_1 with some linear function $\lambda(z)$.

VII. The multiplication formulas for $\wp^2 z$ in the lemniscatic case

We consider here the case in which $h = 3$, and in which, for some $i \geqq i_1$, one of the numbers α_i, β_i, γ_i is 2, and the other two 4. Let, for instance, $\alpha_i = 2$, and $\beta_i = \gamma_i = 4$.

We take first the case of $k = 3$, in which φ_i^{-1} and ψ_i^{-1} have the common critical points a_i, b_i, c_i. As before, we represent the orders of the substitutions which the branches of ψ_i^{-1} undergo at these points by α_i', β_i', γ_i', respectively.

We shall prove that $\alpha_i' = 2$, and that, when $s > 4$, $\beta_i' = \gamma_i' = 4$. When $s = 4$, one of β_i' and γ_i' is 2, and the other is 4. We cannot have $s = 3$.

First let r be odd. Since α_i, β_i, γ_i are divisors of 4, φ_i^{-1} must have at least one simple point at each of a_i, b_i, c_i. If α_i' were not 2, or if β_i' and γ_i' were not divisors of 4, φ_i^{-1} would have A-points in addition to the three simple points.

Suppose that one of β_i' and γ_i' is 2 rather than 4. Let it be γ_i', for instance. Unless $s \leqq 6$, ψ_i^{-1} will surely have more than three A-points at c_i. If $s = 6$, ψ_i^{-1} must have three simple branch points c_i. This leads to the result that φ_{i+1}^{-1} has three critical points at which its branches are permuted in pairs, a situation proved impossible in § V. If $s = 5$, ψ_i^{-1} would have at least three A-points at c_i, and at least one at a_i. If $s = 3$, ψ_i^{-1} would have at least two A-points at c_i, and at least one at each of a_i and b_i.

Thus, when $\gamma_i' = 2$, we have $s = 4$. We must have $\beta_i' = 4$, for the case of $\alpha_i' = \beta_i' = \gamma_i' = 2$ is known to be impossible.

When $s = 4$, it is necessary, since $\beta_i' = 4$, that ψ_i^{-1} have two simple points, either at a_i, or at c_i. They must be at c_i, otherwise ψ_i^{-1} would have four A-points.

We take now the case in which r is even. It is evident that $r > 4$, so that r is at least 6.

First we prove that $\alpha_i' = 2$. If $\alpha_i' > 2$, r must be 6, and φ_i^{-1} must have three simple branch points at a_i. As φ_i^{-1} has branch points of order 3 at b_i and at c_i, φ_i^{-1} must have two simple points at b_i or at c_i, or the sum of its indices would be 11 instead of 10. Thus, φ_i^{-1} would have at least five A-points. Hence $\alpha_i' = 2$.

As $r \geqq 6$, φ_i^{-1} has at least four points at b_i and c_i. Then either β_i' or γ_i' must be a divisor of 4. Let it be β_i', for instance.

Let $\beta_i' = 2$. The very proof used above for r odd shows that $s = 4$. We know that γ_i' cannot equal 2 in this case, and it will be seen below that $\gamma_i' \neq 3$. Hence, if it were possible for s to be 4, we would have $\gamma_i' = 4$. This information will be used below in proving that either r or s must be odd; we shall know thus that β_i' cannot be 2.

Suppose that $s>4$, so that $\beta'_i=4$. By (9), we must have $\gamma'_i=3$ or 4. We show that γ'_i cannot be 3 for any value of s. If γ'_i were 3, at least two of the orders $\alpha'_{i+1}, \beta'_{i+1}, \gamma'_{i+1}$ of the substitutions at the critical points of ψ_{i+1}^{-1} must equal 3. The orders $\alpha_{i+1}, \beta_{i+1}, \gamma_{i+1}$, at the critical points of φ_{i+1}^{-1}, are certainly divisors of 4. They cannot all be 4. If two of them are 2, we see from § V that two of α'_{i+1}, etc. must be 2. If one of α_{i+1}, etc. is 2 and the other two 4, the argument above shows that two of α'_{i+1}, etc. are divisors of 4. Hence $\gamma'_i=4$.

We shall now examine the Riemann surfaces of φ_i^{-1} and ψ_i^{-1}. It will suffice to deal with φ_i^{-1}.

Suppose first that r is odd. Then the indices of φ_i^{-1} at a_i, b_i, c_i are not greater, respectively, than

$$\frac{r-1}{2}, \quad \frac{3(r-1)}{4}, \quad \frac{3(r-1)}{4}.$$

As the sum of the three indices is $2r-2$, the upper bounds just given must be the actual values of the indices. Thus φ_i^{-1} has one simple point and $(r-1)/2$ simple branch points at a_i, and one simple point and $(r-1)/4$ branch points of order 3 at b_i and at c_i.

Let r now be even. We shall prove that φ_i^{-1} has two simple points, either at b_i or at c_i. We know that $\alpha'_{i+1}, \beta'_{i+1}, \gamma'_{i+1}$ are divisors of 4. They cannot all be 2, as we have seen several times. Thus one of them at least must be 4. This is possible only if φ_i^{-1} has either at b_i or at c_i a point whose order is prime to 4. Such a point has to be a simple point if β_i and γ_i equal 4. Also, since r is even, the simple points of φ_i^{-1} must come in pairs.

Suppose that the two simple points are at c_i. If $r\equiv 2$, mod 4, there must be a simple branch point at b_i. If $r\equiv 0$, mod 4, there must be a simple branch point at c_i, in addition to the two simple points. The two simple points and the simple branch point are the A-points of φ_i^{-1}.

We show now that either r or s is odd. Suppose that both are even. By what precedes, we may suppose that ψ_i^{-1} has two simple points at c_i. Thus φ_{i+1}^{-1} would have two critical points with similar substitutions of order 4, whereas the preceding paragraph shows that the substitutions cannot be similar when r is even.

We consider now the case of $k=2$. We must have $s=3$, or $s=2$. If s were 3, ψ_i^{-1} would have an A-point at each of its critical points, which would be points of order 3, and three A-points at that critical point of φ_i^{-1} which is not a critical point of ψ_i^{-1}. Hence, $s=2$. If the critical points of ψ_i^{-1} were b_i and c_i, they would be A-points of ψ_i^{-1} and ψ_i^{-1} would also

have two A-points at a_i. Thus we may assume that the critical points of ψ_i^{-1} are a_i and c_i.

We shall show that, when $k = 2$, r is odd. Suppose that r is even. If ψ_{i+1}^{-1} is to have critical points, φ_i^{-1} must have two simple points, either at a_i or at c_i. Suppose that they are at a_i. Consider the relation

$$F = \varphi_i\,\psi_{i+1} = \psi_i\,\varphi_{i+1}.$$

It follows from $F = \varphi_i\,\psi_{i+1}$ that among the values of F^{-1} at a_i are the two values which ψ_{i+1}^{-1} assumes as its critical points. One of these values is the affix of a critical point of φ_{i+2}^{-1} with a substitution of order 2, and hence the affix of a critical point of ψ_{i+2}^{-1}. Now, from $F = \psi_i\,\varphi_{i+1}$, since the affixes of the critical points of ψ_{i+2}^{-1} are values of φ_{i+1}^{-1} at its simple points, it follows that the value of ψ_i^{-1} at a_i is the affix of a critical point of φ_{i+1}^{-1}, a falsity.

If the two simple points are at c_i, φ_{i+1}^{-1} will have two simple points at its critical point with substitution of order 2, something just seen to be impossible.

Thus, when $k = 2$, r is odd, so that φ_i^{-1} has one simple point at each of a_i, b_i, c_i. Furthermore, it is easy to show, by the method already frequently used, that the value a_{i+1} which φ_i^{-1} assumes at its simple point at a_i is the value of ψ_i^{-1} at c_i, and that the values b_{i+1} and c_{i+1} which φ_i^{-1} assumes at its simple points at b_i and c_i are the two values of ψ_i^{-1} at b_i.

Suppose that r is odd. If s is odd, φ_i^{-1} and ψ_i^{-1} have the same value, a_{i+1}, at their simple points at a_i, and the same pair of values b_{i+1} and c_{i+1} at their simple points at b_i and c_i. If s is even, ψ_i^{-1} takes the values b_{i+1} and c_{i+1} at its simple points, and the value a_{i+1} at its simple branch point which is an A-point.

In what remains to be done, it is unnecessary to use the condition that $r \geqq s$. Accordingly, we work under the legitimate and convenient assumption that r is odd.

The details from this point on are entirely analogous to the corresponding details in the three cases already treated. It is the easiest matter to prove that σ_i^{-1} has no critical points other than $a_{i+1}, b_{i+1}, c_{i+1}$, and that the orders of the substitutions which its branches undergo at those points are divisors of 2, 4 and 4 respectively. Also we work back readily to Φ^{-1} and Ψ^{-1}. These have no critical points other than three certain points a_0, b_0, c_0, at which their branches undergo substitutions whose orders are divisors of 2, 4 and 4 respectively. If the degree of either permutable function is odd, the branches of its inverse are permuted in pairs at a_0, except one which is uniform and has the value a_0. At b_0 and c_0 its branches are permuted in fours.

except that, both at b_0 and at c_0, there is a place on the surface of the inverse where the inverse is uniform, and the values of the inverse at these places are (as a pair) b_0 and c_0. If the degree is even, the branches are permuted in pairs at a_0. Either at b_0 or at c_0, there are two uniform branches whose values are b_0 and c_0, and a simple branch point, where the value assumed is c_0.

To identify Φ and Ψ, we choose any number ω, different from zero, and construct $\wp(z\,|\,\omega, i\omega)$. It is well known that in this lemniscatic case, $e_2 = 0$, and $e_3 = -e_1$. We consider now $\wp^2 z$. Where it assumes the value ∞, namely, at the points congruent to the origin, it assumes it four times. Similarly, the value 0 is assumed only at the points congruent to ω_2, and then four times, while the value e_1^2 is assumed twice at all points congruent to ω_1 and ω_3. There are no values other than ∞, 0, and e_1^2 which are assumed more than once by $\wp^2 z$ at any point.

We now take a linear function $\lambda(z)$ such that

$$\lambda(a_0) = e_1^2, \qquad \lambda(b_0) = 0, \qquad \lambda(c_0) = \infty,$$

and deal with

$$\Phi_1 = \lambda\,\Phi\,\lambda^{-1}, \qquad \Psi_1 = \lambda\,\Psi\,\lambda^{-1}.$$

Precisely as in the preceding section, we find that

$$\wp^2(az+b) = \Phi_1(\wp^2 z), \qquad \wp^2(cz+d) = \Psi_1(\wp^2 z),$$

where

$$2a\omega_i \equiv 0, \quad 2c\omega_i \equiv 0 \quad (\mathrm{modd}\; 2\omega_1, 2\omega_3) \quad (i = 1, 3).$$

To determine b, we note that, in the lemniscatic case, $\wp^2 iz = \wp^2 z$. Hence, if

$$z_1 \equiv iz_2 \qquad (\mathrm{modd}\; 2\omega_1, 2\omega_3),$$

we must have,

$$az_1 + b \equiv iaz_2 + ib \qquad (\mathrm{modd}\; 2\omega_1, 2\omega_3).$$

Multiplying the first congruence through by a, and subtracting the result from the second, we have

$$b(1-i) \equiv 0 \qquad (\text{modd } 2\omega_1, 2\omega_3).$$

A similar condition holds for d. Furthermore, as in the preceding cases

$$(a-1)d \equiv (c-1)b \qquad (\text{modd } 2\omega_1, 2\omega_3).$$

VIII. The multiplication formulas for $\wp' z$ in the equianharmonic case

We take, for $h = 3$, the case in which, for some $i \geqq i_1$, $\alpha_i = \beta_i = \gamma_i = 3$.

We cannot have $r \equiv 2$, mod 3, for in that case, the sum of the indices of φ_i^{-1} could be at most $2r-4$.

If $r \equiv 1$, mod 3, there is one simple point at each critical point of φ_i^{-1}, and the other points are all of order 3.

If $r \equiv 0$, mod 3, there are three simple points at one of the critical points, and none at either of the others.

Let $k = 3$. Then $\alpha_i' = \beta_i' = \gamma_i' = 3$, else φ_i^{-1} would have more A-points than its three simple points. The surface of ψ_i^{-1} is of one of the types described above.

If $k = 2$, we must have $s = 3$, for if s were 2, ψ_i^{-1} would have four A-points. When $k = 2$, we must have $r \equiv 1$, mod 3, and it can be shown, as in the preceding sections, that φ_i^{-1} and ψ_i^{-1} assume the same three values at their simple points.

It is most easy now to introduce σ_i and to work back to Φ and Ψ. The inverses of these two functions have no critical points other than certain three points a_0, b_0, c_0. On the Riemann surface of the inverse of either function, there are precisely three places at a_0, b_0, c_0, at which the inverse is uniform, and the values of the inverse at these places are a_0, b_0, c_0.

To identify Φ and Ψ, we construct $\wp(z \mid \omega, e^{\pi i/3}\omega)$ where ω is any number different from zero. As $\wp(z \mid \omega_1, \omega_3)$ is a homogeneous function of degree -2 in z, ω_1 and ω_3, and as $\wp(z \mid \omega, e^{\pi i/3}\omega)$ is identical with $\wp(z \mid e^{2\pi i/3}\omega, -\omega)$, we have, in the present case,

$$\wp e^{\frac{2\pi i}{3}} z = e^{\frac{2\pi i}{3}} \wp z.$$

Differentiating, we find

$$\wp' e^{\frac{2\pi i}{3}} z = \wp' z, \qquad \wp'' e^{\frac{2\pi i}{3}} z = e^{\frac{4\pi i}{3}} \wp'' z,$$

$$\wp''' e^{\frac{2\pi i}{3}} z = e^{\frac{2\pi i}{3}} \wp''' z.$$

Hence, if

$$e^{\frac{2\pi i}{3}} z \equiv z \qquad (\text{modd } 2\omega,\ 2e^{\frac{\pi i}{3}}\omega), \tag{25}$$

we will have, except for $z \equiv 0$, when $\wp' z = \infty$,

$$\wp'' z = \wp''' z = 0. \tag{26}$$

We consider the following two solutions of (25):

$$z_1 = \frac{2\omega}{1 - e^{\frac{2\pi i}{3}}}, \qquad z_2 = \frac{2\omega + 2e^{\frac{\pi i}{3}}\omega}{1 - e^{\frac{2\pi i}{3}}}.$$

We cannot have $z_1 \equiv z_2$, modd 2ω, $2e^{\frac{\pi i}{3}}\omega$, for

$$z_2 - z_1 = 2\omega \frac{e^{\frac{\pi i}{3}}}{1 - e^{\frac{2\pi i}{3}}},$$

and the modulus of the second member of the last congruence is less than that of 2ω, the period of smallest modulus. We may suppose that z_1 and z_2 lie within the same parallelogram.

From (26), it follows that the values $\wp'(z_1)$ and $\wp'(z_2)$ are each assumed at least three times by $\wp'(z)$ at z_1 and z_2. But since $\wp''(z)$ vanishes just four times in a parallelogram, these values are assumed precisely three times each. Furthermore, no other value, except ∞, is ever assumed more than once at a point. Also, as $\wp'(z)$ assumes every value just three times in a parallelogram, $\wp'(z_1)$ and $\wp'(z_2)$ are not equal.

We now take a linear $\lambda(z)$ such that

$$\lambda(a_0) = \infty, \qquad \lambda(b_0) = \wp'(z_1), \qquad \lambda(c_0) = \wp'(z_2),$$

and let

$$\Phi_1 = \lambda\, \Phi\, \lambda^{-1}, \qquad \Psi_1 = \lambda\, \Psi\, \lambda^{-1}.$$

We find

$$\wp'(az + b) = \Phi_1(\wp' z), \qquad \wp'(cz + d) = \Psi_1(\wp' z),$$

where

$$2\, a\, \omega = 0, \qquad 2\, c\, \omega = 0 \qquad \left(\operatorname{modd} 2\,\omega,\ 2\, e^{\frac{\pi i}{3}}\,\omega\right),$$

$$\left(1 - e^{\frac{2\pi i}{3}}\right) b = 0, \quad \left(1 - e^{\frac{2\pi i}{3}}\right) d = 0 \quad \left(\operatorname{modd} 2\,\omega,\ 2\, e^{\frac{\pi i}{3}}\,\omega\right),$$

and

$$(a - 1)\, d = (c - 1)\, b \qquad \left(\operatorname{modd} 2\,\omega,\ 2\, e^{\frac{\pi i}{3}}\,\omega\right).$$

IX. The multiplication formulas for $\wp^3 z$ in the equianharmonic case

Referring to (9), we find that the only case still to be treated is that, under $h = 3$, in which, for some $i \geqq i_1$, one and only one of $\alpha_i, \beta_i, \gamma_i$ equals 2, and a second equals 3. We let $\alpha_i = 2$, $\beta_i = 3$, $\gamma_i \neq 2$. We may assume that $\alpha_j = 2$, $\beta_j = 3$, $\gamma_j \neq 2$ for every $j > i$, for the work of the preceding sections shows that a different set of values of α_j, etc., would imply other values than those assumed for α_i, etc.

We prove first that $k = 3$. If k were 2, we would have $s = 3$ or $s = 2$. If s were 3, ψ_i^{-1} would have three simple points at that critical point of φ_i^{-1} which is not a critical point of ψ_i^{-1}. This leads to the contradiction that $\alpha_{i+1} = \beta_{i+1} = \gamma_{i+1}$. Suppose that $s = 2$. Then a_i must be a critical point of ψ_i^{-1}, for if the critical points of ψ_i^{-1} were b_i and c_i, ψ_i^{-1} would have an A-point at each of these two points, in addition to two A-points at a_i. The only way in which α_{i+1} can equal 2 is for c_i to be the second critical point of ψ_i^{-1}, and for γ_i to equal 4. Then $\alpha_{i+1} = 2$, $\beta_{i+1} = \gamma_{i+1} = 3$, which is impossible, for the argument just presented shows that one of β_{i+1} and γ_{i+1} must be 4, as γ_i is. Thus $k = 3$.

We must have, of course, $3 \leqq \gamma_i \leqq 6$. We shall show that $\gamma_i = 3$ when $r = 4$, and that otherwise $\gamma_i = 6$.

Suppose first that $\gamma_i = 3$. We cannot have $r \equiv 2$, mod 3, for φ_i^{-1} would have four simple points at b_i and c_i, which would be A-points. Suppose that $r \equiv 0$, mod 3. If there are no simple points at b_i or at c_i, the sum of the indices of φ_i^{-1} at b_i and c_i is $4r/3$. Hence the index at a_i is $2r/3 - 2$. This means, since φ_i^{-1} has only simple branch points at a_i, that

$$\frac{4r}{3} - 4 \leqq r,$$

or $r \leqq 12$. If $r = 12$, φ_i^{-1} has six simple branch points at a_i. Hence, if φ_i^{-1} has an A-point at a_i, it has six, and if it has one at b_i (or at c_i), it has four. If $r = 9$, there is a simple point and four simple branch points at a_i. The simple point must be the only A-point at a_i, else there would be five. But if there were an A-point at b_i (or at c_i) there would be three, which is impossible because of the A-point at a_i. When $r = 6$, there are two simple points at a_i, and the impossibility follows as in the preceding cases. In the case in which $r \equiv 0$, mod 3, and in which there are simple points at b_i, or c_i, we see that there must be just three simple points, either at b_i or at c_i. This would require that the index of φ_i^{-1} at a_i be $2r/3$, which is impossible, since φ_i^{-1} has only simple branch points at a_i.

Thus, when $\gamma = 3$, we have $r \equiv 1$, mod 3, so that there is a simple point at b_i and at c_i. The sum of the indices at b_i and c_i is $4(r-1)/3$, so that the index at a_i is $2(r-1)/3$. Then

$$\frac{4(r-1)}{3} \leqq r,$$

so that $r = 4$. There are two simple branch points at a_i.

In the case of $r = 4$, we must have $s = 4$, or $s = 3$. The branches of ψ_i^{-1} must undergo at a_i a substitution of order 2, else φ_i^{-1} would have two A-points at a_i, in addition to the two simple points at b_i and c_i. Then we cannot have $s = 4$, for in that case, either none or two of α_{i+1} etc. would equal 2, according as ψ_i^{-1} had none or two simple points at a_i. Thus $s = 3$, and ψ_i^{-1} has a simple point and a simple branch point at a_i. At either b_i or c_i, ψ_i^{-1} must have a branch point of order 2, else φ_i^{-1} would have four A-points. Let it be at b_i. Then ψ_i^{-1} has a simple point and a simple branch point at c_i.

We show now that the values 4 and 5 are impossible for γ_i, so that $\gamma_i = 6$ when $r > 4$.

We may suppose that $\alpha_{i+1} = 2$, $\beta_{i+1} = 3$. We shall show first that if γ_i were 4 or 5, γ_{i+1} would equal γ_i. This will follow as soon as we show

that $\gamma_{i+1} \neq 3$, for, from the way in which α_{i+1} etc. depend upon α_i etc., it is clear that γ_{i+1} cannot exceed 4 if $\gamma_i = 4$, and that γ_{i+1} cannot be 4, or more than 5, if $\gamma_i = 5$.

That $\gamma_{i+1} \neq 3$ when $r > 4$ was proved above. Let $r = 4$. We have to consider the possibility of $\gamma_i = 4$. In that case, φ_i^{-1} must have two simple points at a_i and one at b_i. That φ_i^{-1} may have no other A-points, it is necessary that $\beta'_i = 3$. Hence, as $s \leqq 4$, ψ_i^{-1} cannot have more than one A-point at b_i. Thus there cannot be two of the numbers α_{i+1} etc. which equal 3, and as $\beta_{i+1} = 3$, $\gamma_{i+1} \neq 3$. This finishes the proof that $\gamma_{i+1} = \gamma$.

When $\gamma_i = 5$, we see directly that for α_{i+1} etc. to equal 2, 3, 5, respectively, ψ_i^{-1} must have one and only one A-point at each of a_i, etc. Also when $\gamma = 4$, ψ_i^{-1} must have one and only one point at c_i whose order is prime to 4, so that s is odd, and ψ_i^{-1} must also have an A-point at a_i.

Thus, if γ_i is either 4 or 5, ψ_i^{-1} has a single A-point at each of a_i, etc. We shall show that these A-points are simple points, and that α'_i etc. are respectively equal to 2, 3 and γ_i.

First, we cannot have $\alpha'_i = \beta'_i = \gamma'_i = 3$, for the work of the preceding section shows that ψ_i^{-1} would have three simple points at a_i, etc., and ψ_i^{-1} would have other A-points, either at a_i, or at c_i. Hence, by (9), one of α'_i, etc., is 2. Now, neither β'_i nor γ'_i can equal 2, else ψ_i^{-1} would have more than one A-point at b_i or c_i respectively. Thus, $\alpha'_i = 2$, so that ψ_i^{-1} has a simple point at a_i. Now ψ_i^{-1} must have more than one point at c_i, else, since it has only simple branch points at a_i, it would have only simple branch points at b_i, and β'_i would equal 2. As ψ_i^{-1} has just one A-point at c_i, the orders of all of its points but one at c_i must be multiples of γ_i. Since $\beta'_i > 2$, we must have $\gamma'_i < 2\gamma_i$, for $\gamma_i > 3$, and the sum of the reciprocals of α'_i etc. is at least unity. Thus $\gamma'_i = \gamma$, and there is a simple point at c_i. By similar reasoning, we find that $\beta'_i = 3$, and that there is a simple point at b_i.

Now, the sum of the indices of ψ_i^{-1} is

$$\frac{s-1}{2} + \frac{2(s-1)}{3} + \frac{(\gamma_i - 1)(s-1)}{\gamma_i},$$

a quantity inferior to $2s-2$, if γ_i is 4 or 5.

We have proved that $\gamma_i = 6$ when it is not 3.

Before discussing the surfaces of φ_i^{-1} and ψ_i^{-1}, for $\gamma_i = 6$, we shall show that $\alpha'_i = 2$ and $\beta'_i = 3$. If α'_i is not two, the only way in which φ_i^{-1} can have fewer than four A-points at a_i is for φ_i^{-1} to have just six branches, which are permuted in three pairs at a_i. This implies, however, since $\gamma_i = 6$, that φ_i^{-1} has simple points at b_i. Hence $\alpha'_i = 2$. Again, if $\beta'_i \neq 3$, every point of φ_i^{-1}

at b_i is an A-point. Hence, $r < 10$, and $r \neq 8$. Also, if r is odd, φ_i^{-1} has a simple point at a_i, so that r cannot be 9 or 7. Finally, if r be 6, φ_i^{-1} must have no simple points at b_i, and this requires that φ_i^{-1} have four simple points at a_i. Hence $\beta_i' = 3$.

In what follows, we shall not use the condition that $r \geqq s$. Thus all results obtained for φ_i^{-1} will hold for ψ_i^{-1}, when $\gamma_i' = 6$.

We shall show first that we cannot have $r \equiv 2$, mod 3. We may assume that $\alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}$ are 2, 3 and 6 respectively. We know then that $\alpha_{i-1}' = 2$, $\beta_{i-1}' = 3$. Now, as φ_i^{-1} has two simple points at b_i if $r \equiv 2$, mod 3, we must have $\gamma_{i-1}' = 3$. But if $r \equiv 2$, mod 3, not all of the points of φ_{i-1}^{-1} at c_{i-1} can have orders which are multiples of 3. This fact, together with the fact that φ_{i-1}^{-1} has two simple points at b_{i-1}, shows that $\alpha_{i-2}' = \beta_{i-2}' = \gamma_{i-2}' = 3$, whereas we must have $\alpha_{i-2}' = 2$.

Thus, $r \equiv 0, 1, 3$ or 4, mod 6. We examine the surface of φ_i^{-1} for each of these cases.

First, suppose that $r \equiv 0$, mod 6. Every point of φ_i^{-1} at b_i is of order 3. Hence the index of φ_i^{-1} at b_i is $2r/3$. At a_i, φ_i^{-1} will have either none or two simple points. Hence its index at a_i is either $r/2$ or $r/2 - 1$. Thus the index of φ_i^{-1} at c_i is either $5r/6 - 2$ or $5r/6 - 1$; we shall see that it has the former value, so that there are no simple points at a_i.

Suppose that φ_i^{-1} has, at c_i, w points of order 6, x points of order 3, y points of order 2 and z simple points. Then

$$6w + 3x + 2y + z = r. \tag{27}$$

If the index at c were $5r/6 - 1$, we would have

$$5w + 2x + y = \frac{5r}{6} - 1. \tag{28}$$

Multiplying through by 6/5 in (28), and subtracting the result from (27), we find

$$\frac{3x}{5} + \frac{4y}{5} + z = \frac{6}{5}.$$

The only solution of this equation in positive integers is $x = 2$, $y = 0$, $z = 0$. This means that φ_i^{-1} must have two points of order 3 at c_i, and that all its

other points at c_i, if such exist, are of order 6. If γ'_i were not 3, φ_i^{-1} would have at least two A-points at c_i, in addition to the two simple points at a_i. But if $\gamma'_i = 3$, φ_i^{-1} will have no A-points other than the two at a_i. Thus there are no simple points at a_i, and we have

$$5w + 2x + y = \frac{5r}{6} - 2,$$

so that

$$\frac{3x}{5} + \frac{4y}{5} + z = \frac{12}{5}.$$

The solution $x = 4$, $y = z = 0$ leads to an absurdity by the argument just given. The only other solution is $x = y = z = 1$.

Hence, when $r \equiv 0$, mod 6, all branches of φ_i^{-1} must be permuted in pairs at a_i, and in triples at b_i. At c_i, there is a simple point, a point of order 2, and one of order 3. If there are other points at c_i, they are of order 6.

We prove now that $\gamma'_i = 6$. By (9), $2 \leqq \gamma'_i \leqq 6$. If γ'_i were 2, 3 or 4, φ_{i+1}^{-1} would have a critical point with at least three simple points which would arise from the critical point of φ_i^{-1} at c_i. This is impossible, for the critical points of φ_{i+1}^{-1} are evidently similar to those of φ_i^{-1}. If γ'_i were 5, α'_{i+1} etc. would have the impossible values 5, 5, 5. Thus, $\gamma'_i = 6$.

In the case of $r \equiv 1$, mod 6, φ_i^{-1} has one simple point at a_i, and one at b_i. We see as above, only with less trouble, that φ_i^{-1} has one simple point at c_i, and that its other points at c_i are of order 6; also that $\gamma'_i = 2, 3$, or 6, according as $s = 3$, $s = 4$ or $s > 4$.

Similarly, when $r \equiv 3$, mod 6, φ_i^{-1} has a single simple point at a_i, and none at b_i. At c_i, it has a simple point, a simple branch point, and all its other points are of order 6. As before, γ'_i is a divisor of 6. It is impossible for s to be 3. When $s = 4$, $\gamma'_i = 3$, and when $s > 4$, $\gamma'_i = 6$.

We suppose finally that $r \equiv 4$, mod 6. At b_i, φ_i^{-1} must have just one simple point, and its index is $2(r-1)/3$. The index at a_i would be $r/2 - 1$ if there were two simple points at a_i, otherwise $r/2$. Hence the index of φ_i^{-1} at c_i is either

$$\frac{5r}{6} - \frac{1}{3} \quad \text{or} \quad \frac{5r}{6} - \frac{4}{3};$$

we shall show that the index has the latter value, so that φ_i^{-1} has no simple points at a_i.

Suppose that the index had the first value. We would have

$$5w+2x+y=\frac{5r}{6}-\frac{1}{3},$$

and using (27), we find

$$\frac{3x}{5}+\frac{4y}{5}+z=\frac{2}{5},$$

which has no solutions. Hence the second value is the true one, and

$$\frac{3x}{5}+\frac{4y}{5}+z=\frac{8}{5}.$$

This equation has the solutions $x=0$, $y=2$, $z=0$, and $x=1$, $y=0$, $z=1$. We show that the solution $y=2$ is impossible. First, γ_i' is a divisor of 6, else φ_i^{-1} would have at least three A-points at c_i in addition to the simple point at b_i. But as all the points of φ_i^{-1} at c_i would be of even order if $y=2$, the only way for α_{i+1}' to equal 2 would be for γ_i' to be divisible by 4. Hence $y \neq 2$. Thus, φ_i^{-1} has one simple point and one point of order 3 at c_i, and its other points at c are of order 6.

If $s=3$, $\gamma_i'=2$. Also, $s=4$ is impossible. Finally, $\gamma_i'=6$ when $s>3$.

We have already called attention to the fact that when $\gamma_i'=6$ the above discussion of the surface of φ_i^{-1} applies also to the surface of ψ_i^{-1}.

The surfaces of φ_i^{-1} and ψ_i^{-1} being recognized, we can use the methods of the foregoing sections to work back to the surfaces of Φ^{-1} and Ψ^{-1}. There are, indeed, several cases to be examined, and diophantine equations like those used above must be employed in discussing the surface of σ_i^{-1}, but no new ideas have to be introduced.

We find that the degrees of Φ^{-1} and Ψ^{-1} are congruent to $0, 1, 3$ or 4, mod 6, and that the inverse of each has three critical points, called below a_0, b_0, c_0. We shall describe Ψ; similar remarks will apply to Φ.

If $n\equiv 0$, mod 6, the branches of Ψ^{-1} are all permuted in pairs at a_0, and all in triples at b_0. At c_0, Ψ^{-1} has one uniform branch with value c_0, one simple branch point at which $\Psi^{-1}=b_0$, and one branch point of order 2 at which $\Psi^{-1}=a_0$. If $n>6$, the remaining branches are permuted in sixes at c_0.

If $n\equiv 1$, mod 6, Ψ^{-1} has one uniform branch at a_0 whose value is a_0, and the other branches of Ψ^{-1} are permuted in pairs at a_0; at b_0, there is a uniform

branch whose value is b_0, and the other branches are permuted in triples; at c_0, there is one uniform branch whose value is c_0, while the other branches are permuted in sixes.

If $n \equiv 3$, mod 6, there is one uniform branch at a_0 whose value is a_0; at c_0 there is one uniform branch whose value is c_0, and one simple branch point for which $\Psi^{-1} = b_0$. If $n > 3$, the other branches are permuted in sixes at c_0.

If $n \equiv 4$, mod 6, Ψ^{-1} has one uniform branch at b_0 whose value is b_0. At c_0, Ψ^{-1} has one uniform branch whose value is c_0, and one branch point of order 2 at which $\Psi^{-1} = a_0$. Also if $n > 4$. the other branches are permuted in sixes at c_0.

To identify Φ and Ψ, we use the function $\wp(z \mid \omega, e^{\pi i/3}\omega)$ of the preceding section. Putting $\wp\,\omega_1 = e_1$, we have, from the homogeneity formulas,

$$e_2 = \wp\,\omega_2 = e^{\frac{2\pi i}{3}} e_1, \qquad e_3 = \wp\,\omega_3 = e^{\frac{4\pi i}{3}} e_1.$$

The values e_i cannot be zero, else $\wp z$ would have six zeros in a parallelogram. Hence, as $\wp' z$ vanishes only when z is a half-period, $\wp z$ vanishes only once wherever it vanishes.

We consider now the function $\wp^3 z$. Wherever it assumes the values e_1^3, 0, and ∞, it assumes them 2, 3 and 6 times respectively. There are no other multiple values.

We take a linear $\lambda(z)$ such that

$$\lambda(a_0) = e_1^3, \qquad \lambda(b_0) = 0, \qquad \lambda(c_0) = \infty,$$

and introduce Φ_1 and Ψ_1. We find

$$\wp^3(az + b) = \Phi_1(\wp^3 z), \qquad \wp^3(cz + d) = \Psi_1(\wp^3 z),$$

where

$$2\,a\,\omega \equiv 0, \qquad 2\,c\,\omega \equiv 0 \qquad (\text{modd}\,2\,\omega,\, 2\,e^{\frac{\pi i}{3}}\,\omega);$$

$$b\left(1 - e^{\frac{\pi i}{3}}\right) \equiv 0, \qquad d\left(1 - e^{\frac{\pi i}{3}}\right) \equiv 0 \qquad (\text{modd}\,2\,\omega,\, 2\,e^{\frac{\pi i}{3}}\,\omega);$$

$$(a-1)d \equiv (c-1)b \qquad (\text{modd}\,2\,\omega,\, 2\,e^{\frac{\pi i}{3}}\,\omega).$$

The second congruence is found from the equation $\wp^3 e^{\pi i/3} z = \wp^3 z$.

X. Permutable functions with a common iterate

The results of the preceding section are based upon the assumption that every function of the sequence (C) is of degree less than m and less than n; that is, that for no i is one of the functions $\sigma_i \varphi_i$, $\sigma_i \psi_i$ a rational function of the other. The removal of this assumption will lead to a new class of permutable pairs of functions.

It will be convenient, in what follows, to represent $\varphi_i \sigma_i$ by Φ_i and $\psi_i \sigma_i$ by Ψ_i. The preceding sections deal with the sequences

$$(29) \qquad \begin{array}{l} \Phi_0,\ \Phi_1,\ \Phi_2,\ \ldots,\ \Phi_i,\ \ldots \\ \Psi_0,\ \Psi_1,\ \Psi_2,\ \ldots,\ \Psi_i,\ \ldots \end{array}$$

Let us suppose now that for $i \geq i_0$ (this i_0 is not to be confused with the i_0 of § III), one of the functions Φ_i, Ψ_i is a rational function of the other. It fixes the ideas to suppose that i_0 is the smallest number of this type. If we assume that $m \geq n$, we will be sure that it is Φ_{i_0} which is a rational function of Ψ_{i_0}. Let $\Phi_{i_0} = \beta_0 \Psi_{i_0}$. The permutability of Φ_{i_0} and Ψ_{i_0} gives

$$\beta_0 \Psi_{i_0} \Psi_{i_0} = \Psi_{i_0} \beta_0 \Psi_{i_0},$$

or

$$\beta_0 \Psi_{i_0} = \Psi_{i_0} \beta_0,$$

so that Ψ_{i_0} is permutable with β_0.

We distinguish the following two cases:

(a) β_0 is of degree greater than unity;

(b) β_0 is linear.

Suppose that we have met Case (a). It will be convenient to replace the pair of symbols Ψ_{i_0} and β_0 by the pair Φ_{10} and Ψ_{10}, and we suppose the new symbols to replace the old in such a way that the degree of Φ_{10} is not less than that of Ψ_{10}. We have thus

$$\Phi_{i_0} = \Phi_{10} \Psi_{10} = \Psi_{10} \Phi_{10},$$

and Ψ_{i_0} is either Φ_{10} or Ψ_{10}.

We deal now with Φ_{10} and Ψ_{10} exactly as we dealt originally with Φ_0 and Ψ_0, and obtain the sequences

$$\Phi_{10},\ \Phi_{11},\ \Phi_{12},\ \ldots,\ \Phi_{1i},\ \ldots,$$

$$\Psi_{10},\ \Psi_{11},\ \Psi_{12},\ \ldots,\ \Psi_{1i},\ \ldots.$$

The following two cases are here possible:

(a) There is no i such that Φ_{1i} is a rational function of Ψ_{1i};

(b) For $i \geqq i_1$, Φ_{1i} is a rational function of Ψ_{1i}.

If Case (b) is at hand, we deal with Φ_{1i_1} and Ψ_{1i_1} exactly as we treated Φ_{i_0} and Ψ_{i_0} above.

The process we are employing leads finally to two sequences of permutable functions, rational and non-linear,

$$(30)\qquad \begin{matrix} \Phi_{p0},\ \Phi_{p1},\ \Phi_{p2},\ \ldots,\ \Phi_{pi},\ \ldots, \\ \Psi_{p0},\ \Psi_{p1},\ \Psi_{p2},\ \ldots,\ \Psi_{pi},\ \ldots, \end{matrix}$$

the common degree of the functions in the first sequence being at least equal to that of the functions in the second, and the two sequences having one of the two following properties:

(I) There is no i such that Φ_{pi} is a rational function of Ψ_{pi};

(II) For $i > i_p$, Φ_{pi} is a *linear* function of Ψ_{pi}.

Case (I), which does not yield any new types of functions, is quickly disposed of.

We know that Φ_{p0} and Ψ_{p0} come from one of the several types of multiplication theorems discussed in the preceding sections. To take an example which is typical, suppose that the periodic function involved is $\cos z$. Then the inverses of the three functions

$$\Phi_{p0}, \qquad \Psi_{p0}, \qquad \Phi_{p0}\,\Psi_{p0}$$

have no critical points other than certain three points a_0, b_0, c_0, at the first two of which their branches are permuted in pairs, except that there are two places on the surface of each inverse at a_0 and b_0 where each inverse is uniform and assumes the values a_0 and b_0; at c_0, the branches of each inverse are permuted in a single cycle, and the single value of each inverse is c_0. Now, since

$$\Phi_{p-1,\,i_{p-1}} = \Phi_{p0}\,\Psi_{p0},$$

and since $\Psi_{p-1, i_{p-1}}$ is either Φ_{p0} or Ψ_{p0}, we can work back to $\Phi_{p-1,0}$, $\Psi_{p-1,0}$, and show by the familiar process that these last two functions come from the multiplication formulas for cos z. Continuing in this fashion, we find that Φ_0 and Ψ_0 are also given by the multiplication formulas for cos$z$.

An examination of all other possibilities shows similarly that every pair of permutable functions which comes under Case (I) is given by one of the multiplication formulas of the preceding sections.

We take now Case (II), in which Φ_{pi_p} is a linear function of Ψ_{pi_p}. For brevity, we represent these two functions by Φ_∞ and Ψ_∞, respectively. We have

$$\Phi_\infty = \beta \Psi_\infty,$$

where $\beta(z)$ is linear.

As above, Ψ_∞ and β are permutable. We have thus to determine the circumstances under which a rational function of degree greater than unity is permutable with a linear function. This question has been solved by Julia,* but it will not hurt to give a brief treatment of it here.

The linear function $\beta(z)$ has either two fixed points or one. In the first case, if $\lambda(z)$ is a linear function which carries the fixed points to 0 and ∞ respectively, $\lambda\beta\lambda^{-1}$ will be of the form εz. In the second case, if $\lambda(z)$ carries the single fixed point to ∞, $\lambda\beta\lambda^{-1}$ will be of the form $z+h$. Thus, if Φ_0 and Ψ_0 and all of the functions of the several sequences are transformed with λ^{-1}, we may suppose that $\beta(z)$ is either εz or $z+h$.

If $\beta(z)$ were of the form $z+h$, with $h \neq 0$, we would have

$$\Psi_\infty(z+h) = \Psi_\infty(z)+h,$$

so that, indicating differentiation with an accent, we find

$$\Psi'_\infty(z+h) = \Psi'_\infty(z).$$

Hence $\Psi'_\infty(z)$ would be periodic, and, being rational, would be a constant. This would require that $\Psi_\infty(z)$ be linear.

Thus $\beta(z)$ must have two fixed points, and

$$\Psi_\infty(\varepsilon z) = \varepsilon \Psi_\infty(z). \tag{31}$$

* Loc. cit., p. 177.

Differentiating, we find

$$\Psi'_\infty(\epsilon z) = \Psi'_\infty(z).$$

If ϵ were not a root of unity, $\Psi'_\infty(z)$ would assume, for each of the distinct points $\epsilon^r z_1 (z_1 \neq 0, \infty; r = 1, 2, \ldots)$, the same value as at z_1. Hence, $\Psi'_\infty(z)$, being rational, would be a constant, and $\Psi_\infty(z)$ would be linear.

Thus, ϵ is a root of unity, let us say, a primitive rth root of unity. We read directly from (31) that $\Psi_\infty(z)/z$ is a rational function of z^r, that is,

$$\Psi_\infty(z) = z R(z^r),$$

where $R(z)$ is a rational function.

Referring now to the definitions, given in the introduction, of the operations of the first and second types, we see immediately that *if the pair of non-linear permutable functions $\Phi(z)$ and $\Psi(z)$ do not come from the multiplication theorems of the periodic functions, there exists a linear $\lambda(z)$ such that $\lambda\Phi\lambda^{-1}$ and $\lambda\Psi\lambda^{-1}$ can be obtained by repeated operations of the first and second types, starting from a pair of functions*

$$z R(z^r), \qquad \epsilon z R(z^r),$$

where $R(z)$ is rational, and where ϵ is a primitive rth root of unity.

As we do not determine all cases in which operations of the first type are possible, the process just described can certainly not be accepted as furnishing a neat characterization of the pairs of functions which come under Case (II). Nevertheless, we shall progress much further in the study of this case. In particular, we shall settle completely the case in which $\Phi(z)$ and $\Psi(z)$ are polynomials.

We begin by proving that *if the pair of permutable functions $\Phi(z)$ and $\Psi(z)$ come under Case* (II), *in particular, if $\Phi(z)$ and $\Psi(z)$ do not come from the multiplication theorems of the periodic functions, some iterate of $\Phi(z)$ is identical with some iterate of $\Psi(z)$.*

We have shown that

$$\Phi_{pi_p} = \epsilon\,\Psi_{pi_p}.$$

We denote the nth iterate of any function $F(z)$ by $F^{(n)}(z)$. As Ψ_{pi_p} is permutable with ϵz, we have

$$\Phi_{pi_p}^{(n)} = \epsilon^n\,\Psi_{pi_p}^{(n)},$$

and, in particular, since $\epsilon^r = 1$,

$$\Phi^{(r)}_{pi_p} = \Psi^{(r)}_{pi_p}. \tag{32}$$

We have regard now to the manner in which the two functions of (32) are obtained from those which precede them in (30). We write

$$\begin{aligned} \Phi_{p,i_p-1} &= \varphi\sigma, & \Psi_{p,i_p-1} &= \psi\sigma, \\ \Phi_{pi_p} &= \sigma\varphi, & \Psi_{pi_p} &= \sigma\psi; \end{aligned} \tag{33}$$

our failure to attach subscripts to σ, φ and ψ causes no confusion. Thus (32) may be written

$$\sigma\varphi \cdots \sigma\varphi = \sigma\psi \cdots \sigma\psi$$

so that, operating on both sides of the last equation with φ, and replacing z by $\sigma(z)$, we have

$$\varphi\sigma\varphi \cdots \sigma\varphi\sigma = \varphi\sigma\psi \cdots \sigma\psi\sigma.$$

As $\varphi\sigma$ and $\psi\sigma$ are permutable, we find

$$\varphi\sigma\varphi \cdots \sigma\varphi\sigma = \psi \cdots \sigma\psi\sigma\varphi\sigma.$$

Removing $\varphi\sigma$ from the beginning of each member of this equation, we have

$$\varphi\sigma \cdots \varphi\sigma = \psi\sigma \cdots \psi\sigma,$$

that is,

$$\Phi^{(r)}_{p,i_p-1} = \Psi^{(r)}_{p,i_p-1}.$$

Continuing thus, we see that the rth iterates of Φ_{p0} and Ψ_{p0} are equal. Now, since

$$\Phi_{p-1,i_{p-1}} = \Phi_{p0}\,\Psi_{p0}$$

we see directly that

$$\Phi_{p-1,\,i_{p-1}}^{(r)} = \Psi_{p-1,\,i_{p-1}}^{(2r)}.$$

There is nothing to prevent us from working back through the sequence which precedes (30) as we did through (30), and thence onward through the earlier sequences. We find thus that some iterate of $\Phi(z)$ is identical with some iterate of $\Psi(z)$.

We turn for a while to the case in which $\Phi(z)$ and $\Psi(z)$ are polynomials. In a paper published a few years ago,* we determined all pairs of polynomials which have an iterate in common, and the results there produced could be used to settle quickly the problem now before us. However, we shall gain a better insight into the nature of the fractional permutable functions which have an iterate in common by studying the polynomial case from the group-theoretic point of view of the present paper, and of our paper *Prime and composite polynomials*.

From a result stated at the bottom of page 54 of the paper just mentioned, it follows that, if $\Phi(z)$ and $\Psi(z)$ are polynomials, we may suppose that the functions in the sequences (29) to (30) inclusive are all polynomials.

We prove now that when $\Phi(z)$ and $\Psi(z)$ are polynomials, then, in (30), Φ_{p0} is a linear function of Ψ_{p0}. Suppose that this is not so, and that the first pair of functions in (30) which are linear functions of each other are Φ_{pi_p} and Ψ_{pi_p}, where $i_p > 0$. We have already seen that if $\Phi(z)$ and $\Psi(z)$ are subjected to a suitable linear transformation (in this case integral), we will have

$$\Phi_{pi_p} = \varepsilon \Psi_{pi_p},$$

or, by (33),

$$\Phi_{pi_p} = \sigma\varphi = \varepsilon\sigma\psi. \tag{34}$$

Now the algorithm which produces the sequence (30) supposes that no rational $\beta(z)$ of degree greater than 1 exists, such that $\varphi = \zeta\beta$, $\psi = \xi\beta$. We have shown, however, (loc. cit., p. 56) that if Φ_{pi_p} has two decompositions of the types shown in (34), in which φ and ψ are of the same degree, φ must be a linear function of ψ. This completes the proof that Φ_{p0} is a linear function of Ψ_{p0}.

* These Transactions, vol. 21 (1920), p. 313.

We may suppose, thus, that

$$\Phi_{p0} = \varepsilon z R(z^r), \qquad \Psi_{p0} = z R(z^r),$$

where ε is an rth root of unity, and where $R(z)$ is a polynomial.*

Hence

$$\Phi_{p-1,i_{p-1}} = \varepsilon \Psi_{p0}^{(2)}, \qquad \Psi_{p-1,i_{p-1}} = \varepsilon_1 \Psi_{p0}, \tag{35}$$

where ε_1, if not equal to ε, is unity.

We shall show now that $\Phi_{p-1,0}$ is a rational function of $\Psi_{p-1,0}$, so that these two functions may be considered to be given by (35). If this were not the case, we would have

$$\Phi_{p-1,i_{p-1}} = \sigma \varphi, \qquad \Psi_{p-1,i_{p-1}} = \sigma \psi, \tag{36}$$

where no non-linear $\beta(z)$ exists such that $\varphi = \sigma\beta$, $\psi = \xi\beta$. From (35), and the second equation of (36), we find

$$\Phi_{p-1,i_{p-1}} = \varepsilon \varepsilon_1^{-1} \Psi_{p0} \Psi_{p-1,i_{p-1}} = \varepsilon \varepsilon_1^{-1} \Psi_{p0} \sigma \psi. \tag{37}$$

Thus, by (37) and the first equation of (36), φ and ψ would determine systems of imprimitivity of the group of the inverse of $\Phi_{p-1,i_{p-1}}$ with not more than one letter in common. For this it would be necessary (loc. cit., p. 57), that the degrees of φ and ψ be prime to each other. This produces the contradiction that the degree of $\Phi_{p-1,i_{p-1}}$ is not divisible by that of $\Psi_{p-1,i_{p-1}}$. Hence, we have

$$\Phi_{p-2,i_{p-2}} = \varepsilon_2 \Psi_{p0}^{(3)}, \qquad \Psi_{p-2,i_{p-2}} = \varepsilon_3 \Psi_{p0}^{(j)},$$

where ε_2 and ε_3 are rth roots of unity, and where j is 1 or 2.

Continuing thus, we find that *if $\Phi(z)$ and $\Psi(z)$ are permutable polynomials (non-linear), which do not come from the multiplication formulas of e^z or* $\cos z$, *there exist a linear $\lambda(z)$, and a polynomial*

$$G(z) = z R(z^r),$$

* The case of $r = 1$ does not require a separate statement, for we may suppose, using a suitable linear transformation, that all polynomials met are divisible by z.

such that

$$\Phi = \lambda^{-1}(\varepsilon_1 G^{(\mu)} \lambda), \qquad \Psi = \lambda^{-1}(\varepsilon_2 G^{(\nu)} \lambda)$$

where ε_1 and ε_2 are rth roots of unity.

Furthermore, this necessary condition for permutability is immediately seen to be sufficient. In fact, if $R(z)$ is any rational function, integral or fractional. the above formulas will give a pair of permutable rational functions.

When we seek explicit formulas for the permutable pairs of fractional functions which do not come from the multiplication theorems of the periodic functions, things do not go through smoothly. For instance, it is not necessary that one of the functions Φ and Ψ should be a rational function of the other, as is always the case for polynomials.

We shall give an example of such a case. Let

$$\varphi(z) = \frac{\varepsilon^2 z^2 + 2}{\varepsilon z + 1}, \qquad \psi(z) = \frac{z^2 + 2}{z + 1}, \qquad \sigma(z) = \frac{z^2 - 4}{z - 1},$$

where ε is a primitive third root of unity. We shall see below that $\Phi = \varphi\sigma$ and $\Psi = \psi\sigma$ are permutable. We observe at present that Φ is not a linear function of Ψ; if it were, φ would be a linear function of ψ, which is not so. because ψ is infinite for $z = -1$ and $z = \infty$, whereas φ is infinite for $z = \infty$, but not for $z = -1$. We have

$$\sigma\varphi = \varepsilon z \frac{z^3 - 8}{z^3 - 1}, \qquad \sigma\psi = z \frac{z^3 - 8}{z^3 - 1}.$$

As $\varphi(z) = \psi(\varepsilon z)$, and as $\sigma\psi(\varepsilon z) = \varepsilon\sigma\psi(z)$, we have

$$\varphi\sigma\psi\sigma = \psi(\varepsilon\sigma\psi\sigma) = \psi\sigma\psi(\varepsilon\sigma) = \psi\sigma\varphi\sigma.$$

This means that Φ and Ψ, which, we repeat, are not linear functions of each other, are permutable.

It can be shown that Φ and Ψ do not come from the multiplication theorems of the periodic functions. We shall escape the calculations connected with this question by modifying the example. Let $\beta(z)$ be any rational function such that $\beta(\varepsilon z) = \varepsilon\beta(z)$, where ε is a primitive third root of unity. We consider the functions

$$\Phi = \varphi\beta\sigma, \qquad \Psi = \psi\beta\sigma.$$

where σ, φ and ψ are the three functions of the second degree used above. As above, Φ is not a linear function of Ψ. Also,

$$\varphi\beta\sigma\psi\beta\sigma = \psi(\varepsilon\beta\sigma\psi\beta\sigma) = \psi\beta\sigma\psi(\varepsilon\beta\sigma) = \psi\beta\sigma\varphi\beta\sigma,$$

so that Φ and Ψ are permutable.

It is easily seen, since $\beta(z)$ is of a very general type, that, by suitably choosing $\beta(z)$, we can make the critical points of Φ^{-1} and of Ψ^{-1} numerous and complicated at pleasure. This means that Φ and Ψ cannot come from the multiplication theorems of the periodic functions, for, if they did, their inverses would have at most four critical points.

In the above example, Φ and Ψ have the same third iterate.

Concerning the pairs of permutable fractional functions which come neither from the multiplication theorems of the periodic functions, nor from the iteration of a function, the only information we have, in addition to the fact that the functions of the pair have an iterate in common, is that contained in the statement on page 443. We think that the example given above makes it conceivable that no great order may reign in this class.

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ON APPROXIMATION BY FUNCTIONS OF GIVEN CONTINUITY*

BY

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1. **Introduction.** Let $f(x)$ be a given function, continuous for $a \leqq x \leqq b$. Let λ be a given constant $\geqq 0$, and let Φ_λ, or, when no confusion will arise, simply Φ, denote the family of all functions $\varphi(x)$ satisfying the condition

$$|\varphi(x_2) - \varphi(x_1)| \leqq \lambda |x_2 - x_1|$$

whenever $a \leqq x_1 \leqq b$, $a \leqq x_2 \leqq b$. This note is concerned with the question of approximating $f(x)$ by means of a function of the family Φ. The discussion is deliberately restricted in scope, for the sake of illustrating the leading ideas with the greatest practicable simplicity. It could be generalized, to a greater or less extent, in any one of a number of different directions.

2. **Existence and uniqueness of the approximating function according to the criterion of least mth powers,** $m \geqq 1$. Let $\varphi(x)$ be an arbitrary function of the family Φ. Such a function will be called for brevity "a function φ". Let m be a given constant $\geqq 1$. Attention will be directed first to the minimizing of the integral

$$\delta = \int_a^b |f(x) - \varphi(x)|^m \, dx.$$

As φ ranges over the family Φ, the value of δ has a lower limit $\gamma \geqq 0$. It will be shown that there is one and just one function φ for which δ attains the value γ. It is understood that the function $f(x)$ and the constants λ and m are held fast.

If $f(x)$ itself belongs to Φ, the value of γ is evidently zero, attained by taking $\varphi = f$, and by no other choice of φ. If f is not a function φ, let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots$ be a set of positive numbers approaching zero, and for each value of k let φ_k be a function φ for which

$$(1) \qquad \delta < \gamma + \varepsilon_k.$$

* Presented to the Society, April 14, 1923.

It may be assumed without loss of generality that

$$\mu \leqq \varphi_k(x) \leqq M \tag{2}$$

throughout (a, b), where μ and M are the minimum and the maximum of $f(x)$ in the interval. For if this is not the case originally, a function φ'_k, belonging to Φ, and satisfying the condition (1), can be defined by taking $\varphi'_k = \varphi_k$ when $\mu \leqq \varphi_k \leqq M$, $\varphi'_k = \mu$ when $\varphi_k \leqq \mu$, and $\varphi'_k = M$ when $\varphi_k \geqq M$. The functions φ_k will be considered chosen so that (2) is fulfilled. Then, being uniformly bounded (because of (2)) and equally continuous (by the definition of Φ), they form a sequence to which the theorem of Ascoli and Arzelà may be applied. There will be a sub-sequence among them which approaches uniformly a limit $\tau(x)$. It is recognized immediately that the uniform limit of a sequence of functions φ is a function φ. On the other hand, the values of δ corresponding to a uniformly convergent sequence of functions φ approach the value of δ corresponding to the limiting function of the sequence. As the limit of δ for the sequence (φ_k), or any sub-sequence chosen from it, is γ, because of (1), it appears that $\tau(x)$ is a function of the family Φ, for which*

$$\int_a^b |f - \tau|^m \, dx = \gamma.$$

Such a function will be called briefly an *approximating function* for $f(x)$, corresponding to the parameters λ and m.

Suppose τ_1 and τ_2 are two functions φ, for each of which $\delta = \gamma$. Let $\tau_3 = \frac{1}{2}(\tau_1 + \tau_2)$. Then τ_3 is also a function φ. Furthermore, $f - \tau_3 = \frac{1}{2}[(f - \tau_1) + (f - \tau_2)]$, and

$$|f - \tau_3|^m \leqq \tfrac{1}{2}[|f - \tau_1|^m + |f - \tau_2|^m],$$

the relation being an actual inequality at any point where $\tau_1 \neq \tau_2$. So if τ_1 and τ_2 are not identical, the value of δ corresponding to τ_3 is less than γ, which is impossible. This proves the uniqueness of the approximating function τ.

The approximating function for given λ and m will be denoted henceforth by $\varphi_{\lambda m}(x)$, instead of $\tau(x)$.

* Somewhat more directly, the existence of a minimum follows almost immediately from a well known theorem in the theory of functions of lines.

The above discussion has assumed that $m > 1$. The proof of the existence of at least one approximating function applies without change if $0 < m \leqq 1$. Examples show* that when $m < 1$ there may be more than one approximating function for a given value of m. In the limiting case $m = 1$, the approximating function is unique once more, though a modified form of demonstration is required.

Suppose, if possible, that r_1 and r_2 are two distinct approximating functions corresponding to the exponent $m = 1$, and let $r_3 = \frac{1}{2}(r_1 + r_2)$. It is still true that

$$|f - r_3| \leqq \tfrac{1}{2}[|f - r_1| + |f - r_2|],$$

but the equality sign is not necessarily ruled out, even at points where $r_1 \neq r_2$. The relation would be an inequality, however, and a contradiction would result, if $f - r_1$ and $f - r_2$ were to take on *opposite signs* at any point. It must be assumed that $(f - r_1)(f - r_2) \geqq 0$ throughout (a, b). This being the case, let a function r_4 be defined at each point of (a, b) as equal to r_1 or r_2, according as $|f - r_1| \leqq$ or $\geqq |f - r_2|$. Then r_4 is continuous, belongs to the family Φ, and gives a better approximation, according to the exponent $m = 1$, than either r_1 or r_2, under the assumption that the latter functions are not identical. Thus a contradiction is obtained in the present circumstances.

3. **Existence and indeterminacy of the approximating function according to the criterion of least maximum error.** For an arbitrary φ, let d be the maximum of $|f - \varphi|$ in the interval $a \leqq x \leqq b$. The values of d corresponding to the various members of the family Φ have a lower limit $\geqq 0$, which may be denoted by e. By reasoning altogether parallel to that of the preceding section, it may be shown that there exists a function r in the family Φ, for which d actually has the value e.

This function r, however, is not in general uniquely determined. For example, let $f(x) = \sqrt{x}$ in the interval $0 \leqq x \leqq 1$, and let $\lambda = 1$. Here the value of e can not be less than $\frac{1}{8}$. For as x goes from 0 to $\frac{1}{4}$, $f(x)$ increases from 0 to $\frac{1}{2}$, while no function φ can increase by more than $\frac{1}{4}$, so that any φ must differ from f by at least $\frac{1}{8}$ either for $x = 0$ or for $x = \frac{1}{4}$. On the other hand, d is actually equal to $\frac{1}{8}$ for the function $r_1(x) = x + \frac{1}{8}$, and for the function $r_2(x)$ which is equal to $x + \frac{1}{8}$ for $0 \leqq x \leqq \frac{1}{8}(3 + 2\sqrt{2})$ and to $\sqrt{x}$

* E. g., let $f(x)$ be defined for $-1 \leqq x \leqq 1$ as equal to -1 when $x \leqq -\varepsilon$, equal to $+1$ when $x \geqq \varepsilon$, and equal to x/ε when $-\varepsilon \leqq x \leqq \varepsilon$. Let $\lambda = 0$, $m = \frac{1}{2}$. The functions φ are constants. The approximating constant C can not be unique unless $C = 0$, for C and $-C$ give the same value to the integral δ. But $C = 0$ does not give the best approximation, when ε is taken arbitrarily small, because the values of δ corresponding to $C = 0$ and $C = \pm 1$ are arbitrarily near to 2 and $\sqrt{2}$ respectively.

for $\frac{1}{8}(3+2\sqrt{2}) \leqq x \leqq 1$ (the number $\frac{1}{8}(3+2\sqrt{2})$ being the larger of the two roots of the equation $x + \frac{1}{8} = \sqrt{x}$), and for infinitely many other functions of the family Φ.

The value of e for given λ will be denoted by d_λ, and any function φ for which $d = d_\lambda$ will be called an approximating function, and represented by the notation $\psi_\lambda(x)$.

4. **Convergence for** $m \to \infty$. Let $d_{\lambda m}$ represent the maximum of $|f(x) - \varphi_{\lambda m}(x)|$ for $a \leqq x \leqq b$. If λ is held fast and m is allowed to become infinite, it will be shown that

$$\lim_{m \to \infty} d_{\lambda m} = d_\lambda. \tag{3}$$

As the functions $\varphi_{\lambda m}$, for fixed λ, are uniformly bounded* and equally continuous, they have one or more limit functions as m becomes infinite, uniformly approached by sequences suitably chosen from among them. When (3) has been established, it will follow that any one of these limit functions is a function ψ_λ solving the problem of § 3.

To prove (3), let ϵ be an arbitrary positive quantity. Suppose that $d_{\lambda m} \geqq d_\lambda + \epsilon$, for a specified value of m. Let x_0 be a value of x for which $|f - \varphi_{\lambda m}| = d_{\lambda m}$. By the assumed continuity of $f(x)$, there exists an $h_1 > 0$ such that

$$|f(x) - f(x_0)| < \frac{\epsilon}{4}$$

for $|x - x_0| < h_1$. On the other hand,

$$|\varphi_{\lambda m}(x) - \varphi_{\lambda m}(x_0)| < \frac{\epsilon}{4}$$

when $|x - x_0| \leqq \epsilon/(4\lambda)$. Let h be the smaller of the numbers h_1 and $\epsilon/(4\lambda)$. This h is independent of m, though it of course depends on λ, which is understood to be held fast throughout the present discussion. For $|x - x_0| \leqq h$,

$$|f(x) - \varphi_{\lambda m}(x)| \geqq d_\lambda + \frac{\epsilon}{2}.$$

* It is clear that $\mu \leqq \varphi_{\lambda m} \leqq M$, for all values of λ, m, and x, either directly from the definition of $\varphi_{\lambda m}$ or from the discussion by which its existence was established in § 2.

This relation then holds throughout an interval of length at least h, even if x_0 happens to coincide with a or b. Consequently

$$(4) \qquad \int_a^b |f(x) - \varphi_{\lambda m}(x)|^m \, dx \geq h\left(d_\lambda + \frac{\varepsilon}{2}\right)^m.$$

But

$$(5) \qquad \int_a^b |f(x) - \psi_\lambda(x)|^m \, dx \leq (b-a)\, d_\lambda^m.$$

As the left-hand member of (4) must be less than or equal to the left-hand member of (5), by the definition of $\varphi_{\lambda m}$, it must be that

$$h\left(d_\lambda + \frac{\varepsilon}{2}\right)^m \leq (b-a)\, d_\lambda^m,$$

a relation which ultimately ceases to hold when m becomes indefinitely large. So the assumed relation $d_{\lambda m} \geq d_\lambda + \varepsilon$ leads to a contradiction, as soon as m is sufficiently large, and it must be that $d_{\lambda m} < d_\lambda + \varepsilon$ for all values of m from a certain point on. As $d_{\lambda m} \geq d_\lambda$ throughout, from the definition of d_λ, the truth of (3) is established.

5. **Convergence for $\lambda = \infty$.** In the case of the approximating functions of § 3, it is almost immediately evident that*

$$\lim_{\lambda \to \infty} \psi_\lambda(x) = f(x),$$

uniformly for $a \leq x \leq b$. For $f(x)$, by reason of its uniform continuity, can be approximated with any desired accuracy by a continuous single-valued function whose graph is a broken line with a finite number of segments, and any function of this character belongs to the family Φ_λ as soon as λ is sufficiently large, while the definition of ψ_λ requires that its maximum error be not greater than that of any other function in Φ_λ.

The corresponding problem of convergence for the approximating functions

* When the approximating function is not unique, ψ_λ may be *any* approximating function corresponding to the value of λ in question.

$\varphi_{\lambda m}$, when m is held fast* and λ is allowed to become infinite, appears to be somewhat less trivial. Let $\omega(h)$ be the modulus of continuity of $f(x)$, the maximum of $|f(x')-f(x'')|$ for $|x'-x''|\leq h$. *It will be shown that*

$$\lim_{\lambda\to\infty}\varphi_{\lambda m}(x)=f(x),$$

uniformly for $a\leq x\leq b$, provided that a positive constant k and a constant $\alpha>1/(m+1)$ exist so that $\omega(h)\leq kh^\alpha$ for $0<h<b-a$. It may be assumed without loss of generality† that $\alpha<1$.

For the purposes of the proof, it will be convenient to examine more closely the approximation of $f(x)$ by means of a broken-line function. Let h have any positive value $\leq b-a$. Let $q(x)$ be a broken-line function defined by the requirement that it shall coincide with $f(x)$ for $x=a$, $a+h$, $a+2h,\ldots,$ $a+nh$, where n is the greatest integer contained in $(b-a)/h$, and shall be equal to $f(b)$ for $x=a+(n+1)h$. In any one interval of length h, neither $f(x)$ nor $q(x)$ can change by more than $\omega(h)$, and consequently

$$|f(x)-q(x)|\leq 2\,\omega(h)$$

for $a\leq x\leq b$.

Let λ have any value $\geq k(b-a)^{\alpha-1}$. For this λ, let a number h be determined so that $\lambda h=kh^\alpha$, that is, let $h=(\lambda/k)^{1/(\alpha-1)}$. The slope of any segment of the graph of the function $q(x)$ defined above can not exceed $\omega(h)/h$, which is less than or equal to λ; $q(x)$ belongs to the family Φ_λ, and $d_\lambda\leq 2\,\omega(h)\leq 2kh^\alpha$. By reason of the assumption that $\alpha<1$, h will approach zero when λ is subsequently allowed to become infinite.

Let x_0 be a value of x for which $|f(x)-\varphi_{\lambda m}(x)|$ attains its maximum value $d_{\lambda m}$. Let β be a positive constant, presently to be specified. For $|x-x_0|\leq h^\beta$,

$$|f(x)-f(x_0)|\leq\omega(h^\beta)\leq kh^{\alpha\beta},\qquad |\varphi_{\lambda m}(x)-\varphi_{\lambda m}(x_0)|\leq\lambda h^\beta=kh^{\alpha+\beta-1},$$

and consequently

$$|f(x)-\varphi_{\lambda m}(x)|\geq d_{\lambda m}-kh^{\alpha\beta}-kh^{\alpha+\beta-1}. \tag{6}$$

* If $m<1$, $\varphi_{\lambda m}$ is understood to represent an arbitrary one of the approximating functions, in case the determination is not unique. With this understanding, the conclusion holds for any value of $m>0$.

† If $\alpha=1$, $f(x)$ itself satisfies a Lipschitz condition, and is identical with $\varphi_{\lambda m}(x)$ for all values of m, as soon as λ is sufficiently large; if $\alpha>1$, $f(x)$ is constant, and the problem becomes still more trivial.

Let it be assumed for the moment that the right-hand member here is positive or zero; the contrary hypothesis will be considered later. If β is held fast, h^β will certainly be less than $b-a$, at any rate for values of λ that are sufficiently large, and (6) will hold throughout an interval of length at least h^β, wherever x_0 may be situated. Then

$$\int_a^b |f(x)-\varphi_{\lambda m}(x)|^m\,dx \geqq h^\beta[d_{\lambda m}-kh^{\alpha\beta}-kh^{\alpha+\beta-1}]^m.$$

Since $d_\lambda < 2kh^\alpha$,

$$\int_a^b |f(x)-\psi_\lambda(x)|^m\,dx \leqq (b-a)(2kh^\alpha)^m.$$

The defining property of $\varphi_{\lambda m}$ then requires that

$$h^\beta[d_{\lambda m}-kh^{\alpha\beta}-kh^{\alpha+\beta-1}]^m < (b-a)(2kh^\alpha)^m.$$

$$d_{\lambda m} < kh^{\alpha\beta}+kh^{\alpha+\beta-1}+2k\cdot\sqrt[m]{b-a}\cdot h^{\alpha-\beta/m}.$$

The last relation has been obtained on the assumption that $d_{\lambda m} \geqq kh^{\alpha\beta} + kh^{\alpha+\beta-1}$, but clearly holds in the contrary case as well. The hypothesis of the theorem to be proved specifies that $\alpha > 1/(m+1)$; if β is taken equal to $m/(m+1)$, all three of the exponents on the right are positive, the entire right-hand member approaches zero as λ becomes infinite, and the uniform convergence of $\varphi_{\lambda m}(x)$ to the value of $f(x)$ is proved.

It is perhaps superfluous here to attempt further refinement of the sufficient condition that has been obtained for convergence, as it seems unlikely that the present method would ever yield anything even remotely approaching a necessary condition.

6. A property of the approximating functions $\varphi_{\lambda m}$. In conclusion, a property will be pointed out which in some degree, though by no means completely, characterizes the functions $\varphi_{\lambda m}$. Roughly expressed, the observation is that $\varphi_{\lambda m}$, in order to hold its title of approximating function, must make the utmost possible use of such freedom of oscillation as is left to it by the Lipschitz condition defining the family Φ_λ, wherever it does not attain coincidence with $f(x)$.

To begin with a simple case, suppose $m = 2$, and suppose, if possible, that there is a constant $\lambda' < \lambda$, such that the quotient $|\varphi_{\lambda_2}(x_2) - \varphi_{\lambda_2}(x_1)| / |x_2 - x_1|$ is less than or equal to λ' throughout (a, b). Let $\theta(x)$ be an arbitrary function for which the corresponding quotient never exceeds unity. Then, if r is a sufficiently small quantity, $\varphi_{\lambda_2}(x) + r\theta(x)$ belongs to Φ_λ, and for $r \neq 0$ can not give so good an approximation as $\varphi_{\lambda_2}(x)$, according to the integral of the square of the error. Let

$$\delta(r) = \int_a^b [f(x) - \varphi_{\lambda_2}(x) - r\theta(x)]^2 dx.$$

Then

$$\frac{d}{dr}\delta(r) = \delta'(r) = -2\int_a^b [f(x) - \varphi_{\lambda_2}(x) - r\theta(x)]\theta(x)dx.$$

Since $\delta(r)$ must have a minimum for $r = 0$, it must be that $\delta'(0) = 0$, that is,

$$\int_a^b [f(x) - \varphi_{\lambda_2}(x)]\theta(x)dx = 0.$$

This relation must hold for every function $\theta(x)$ which satisfies a Lipschitz condition with coefficient 1. But a θ can be chosen to violate it, if there is a point in (a, b) at which $f - \varphi_{\lambda_2} \neq 0$. So the approximating function $\varphi_{\lambda_2}(x)$ can not satisfy a Lipschitz condition with coefficient smaller than λ, unless it is identical with $f(x)$ throughout.

More generally, it would be easy to show, by an argument similar in character though different to some extent in detail, that if m has any value > 0, if x_0 is any value of x in (a, b) for which $f - \varphi_{\lambda m} \neq 0$, and if (a', b') is any interval containing x_0 as interior or end point, the quotient $|\varphi_{\lambda m}(x_2) - \varphi_{\lambda m}(x_1)| / |x_2 - x_1|$ must take on values arbitrarily near to λ in (a', b'). The underlying fact, however, appears to be that contained in the following somewhat less elementary theorem:

Let E_1 be the set of points in (a, b) where $\varphi_{\lambda m}(x) = f(x)$, E_2 the set where $\varphi_{\lambda m}$ has a derivative equal to $\pm\lambda$, and E_0 the set where neither of these conditions is satisfied. Then the set E_0 is of measure zero.

As $\varphi_{\lambda m}$ is absolutely continuous,* it is known at the outset that it has a derivative almost everywhere. Assume that the theorem is not true, that is,

* Cf., e. g., de la Vallée Poussin, *Intégrales de Lebesgue*, Paris, 1916, pp. 76 ff.

that $m E_0 > 0$. Let $\varphi'_{\lambda m}$ be equal to the derivative of $\varphi_{\lambda m}$ where the derivative exists, and equal to zero elsewhere. There must be a positive number η such that $m E_3 > 0$, if E_3 consists of those points of E_0 at which $|f - \varphi_{\lambda m}| > \eta$. Let E'_3 and E''_3 be the parts of E_3 where $f - \varphi_{\lambda m} > \eta$ and $f - \varphi_{\lambda m} < -\eta$ respectively. At least one of these sets must have a measure greater than zero, and it may be assumed for definiteness that $m E'_3 > 0$. The set of *all* points where $f - \varphi_{\lambda m} > \eta$ is an open set (since the difference in question is a continuous function of x), and consists of a finite number or an enumerable infinity of open intervals. There must be at least one of these intervals which by itself contains a part of E'_3 of measure greater than zero. Let (a', b') be such an interval, and let E_4 be the part of E'_3 contained in (a', b'). Let $\theta'(x)$ be a function equal to $\lambda - |\varphi'_{\lambda m}(x)|$ (and therefore positive) at the points of E_4, and equal to zero everywhere else, and let

$$\theta(x) = \int_a^x \theta'(x)\, dx,$$

the integral being taken in the sense of Lebesgue, if necessary. Then $\theta(x)$ is identically zero for $a \leq x \leq a'$, increases to a positive value A for $x = b'$ (since it is the integral of a function which is never negative, and which is actually positive throughout a set of positive measure between a' and b') and is constantly equal to A for $b' \leq x \leq b$. Being continuous, θ must take on the value $\frac{1}{2} A$ for at least one point $x = c'$ between a' and b'. Let

$$\theta'_1(x) = \theta'(x) \text{ for } a \leq x \leq c', \quad \theta'_1(x) = -\theta'(x) \text{ for } c' < x \leq b,$$

and let

$$\theta_1(x) = \int_a^x \theta'_1(x)\, dx.$$

The new function $\theta_1(x)$, likewise continuous, is identically zero for $a \leq x \leq a'$ and for $b' \leq x \leq b$, is never negative, and is actually positive for $x = c'$, and hence throughout some interval contained between a' and b'.

The function $\varphi_{\lambda m}(x) + r\,\theta_1(x)$ remains in the family Φ_λ for all values of r belonging to the interval $0 \leq r \leq 1$, because

$$|\varphi'_{\lambda m}(x) + r\,\theta'_1(x)| \leq |\varphi'_{\lambda m}(x)| + r\,|\theta'_1(x)| \leq \lambda.$$

and

$$[\varphi_{\lambda m}(x_2) + r\theta_1(x_2)] - [\varphi_{\lambda m}(x_1) + r\theta_1(x_1)]$$

$$= \left| \int_{x_1}^{x_2} (\varphi'_{\lambda m} + r\theta'_1)\, dx \right| < \lambda\, |x_2 - x_1|.$$

At the same time, $|\theta'_1| \leq \lambda$, and $|\theta_1| \leq \lambda(b-a)$, throughout (a, b). If r is chosen to satisfy the requirements

$$0 < r < 1, \qquad r < \frac{\eta}{\lambda(b-a)},$$

it is certain that $f - \varphi_{\lambda m} - r\theta_1$ is identical with $f - \varphi_{\lambda m}$ for $a \leq x \leq a'$ and for $b' \leq x \leq b$, that

$$0 < f - \varphi_{\lambda m} - r\theta_1 \leq f - \varphi_{\lambda m}$$

throughout the interior of (a', b'), and that the last relation is an actual inequality at the point $x = c'$ and throughout an interval containing this point. This means that the function $\varphi_{\lambda m} + r\theta_1$, belonging to Φ_λ, gives a smaller value to the integral δ than the function $\varphi_{\lambda m}$, and the contradiction which was needed to establish the theorem has been obtained.

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